

DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN

JUL 19 1979

University of Wisconsin
Department of Statistics
1210 West Dayton Street
Madison, Wisconsin 53706

TECHNICAL REPORT NO. 557

March 1979

CONVERGENCE RATES OF "THIN PLATE" SMOOTHING
SPLINES WHEN THE DATA ARE NOISY

Grace Wahba

JUL 19 1979

CONVERGENCE RATES OF "THIN PLATE" SMOOTHING SPLINES WHEN THE DATA ARE NOISY
(Preliminary report)

Grace Wahba
Department of Statistics
University of Wisconsin
Madison, Wisconsin 53705 U.S.A.

Suppose $u \in H^{2m}(\Omega)$ and certain other conditions are satisfied. Then
$$\min_{\lambda} ER(\lambda) = O(n^{-4m/(4m+d)}).$$

Abstract

We study the use of "thin plate" smoothing splines for smoothing noisy d dimensional data. The model is

$$z_i = u(t_i) + \varepsilon_i, \quad i = 1, 2, \dots, n,$$

where u is a real valued function on a closed, bounded subset Ω of Euclidean d -space and the ε_i are random variables satisfying $E\varepsilon_i = 0$, $E\varepsilon_i \varepsilon_j = \sigma^2 \delta_{ij}$, $i, j = 1, \dots, n$. The z_i are observed. It is desired to estimate u , given $z_1, \dots, z_n$. u is only assumed to be "smooth", more precisely we assume that u is in the Sobolev space $H^m(\Omega)$ of functions with partial derivatives up to order m in $L_2(\Omega)$, with $m > d/2$. u is estimated by $u_{n,m,\lambda}$, the restriction to Ω of $\tilde{u}_{n,m,\lambda}$, where $\tilde{u}_{n,m,\lambda}$ is the solution to: Find $\tilde{u}$ (in an appropriate space of functions on R^d) to minimize

$$\frac{1}{n} \sum_{i=1}^n (\tilde{u}(t_i) - z_i)^2 + \lambda \sum_{i_1, \dots, i_m=1}^n \left(\frac{\partial^m \tilde{u}}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_m}} \right)^2 dx_1 dx_2 \dots dx_d.$$

This minimization problem is known to have a solution for $\lambda > 0$, $m > \frac{d}{2}$, $n \geq \binom{m+d-1}{d}$, provided the $t_1, \dots, t_n$ are "unisolvant". We consider the integrated mean square error

$$R(\lambda) = \frac{1}{|\Omega|} \int_{\Omega} (u_{n,m,\lambda}(t) - u(t))^2 dt, \quad |\Omega| = \int_{\Omega} dt,$$

and $ER(\lambda)$, as $\{t_i\}_{i=1}^n$ become dense in Ω . An estimate of λ which asymptotically minimizes $ER(\lambda)$ can be obtained by the method of generalized cross-validation. In this paper we give plausible arguments and numerical evidence supporting the following conjectures:

Suppose $u \in H^m(\Omega)$. Then

$$\min_{\lambda} ER(\lambda) = O(n^{-2m/(2m+d)}).$$

1. Introduction

Consider the model

$$z_i = u(t_i) + \epsilon_i, \quad i = 1, 2, \dots, n \quad (1.1)$$

where u is some "smooth" function on Ω , a closed, bounded subset of R^d , and the $\{\epsilon_i\}$ are independent, zero mean random variables with common unknown variance σ^2 . The $t_1, \dots, t_n$ are in Ω , and $z = (z_1, \dots, z_n)'$ is observed. It is desired to estimate u nonparametrically from z .

Our estimate $u_{n,m,\lambda}$ for u will be obtained as follows:

Let $\tilde{u}_{n,m,\lambda}$ be the solution the the following minimization problem: Find $\tilde{u} \in \bar{X}$ to minimize

$$\frac{1}{n} \sum_{i=1}^n (\tilde{u}(t_i) - z_i)^2 + \lambda \sum_{i_1, \dots, i_m=1}^d \int_{R_d} \left(\frac{\partial^m \tilde{u}}{\partial x_{i_1} \dots \partial x_{i_m}} \right)^2 dx_1 \dots dx_d.$$

For example, when $d=2$, $m=2$, the second or "smoothness penalty" term becomes

$$\lambda \int_{R^2} (\tilde{u}^2_{x_1 x_1} + 2\tilde{u}^2_{x_1 x_2} + \tilde{u}^2_{x_2 x_2}) dx_1 dx_2,$$

which is the bending energy of a thin plate. The space $\bar{X}$ is the "Beppo Levi" space

$$\bar{X} = \{ \tilde{u} \in \mathcal{D}', \frac{\partial^{\alpha_1 + \dots + \alpha_d} \tilde{u}}{\partial x_1 \dots \partial x_d} \in L_2(R^d), \text{ for } \alpha_1 + \dots + \alpha_d = m \}$$

where $\mathcal{D}'$ is the dual of the Schwartz space $\mathcal{D}$ of infinitely differentiable functions with compact support. See Meinguet (1978,1979) for further details. $u_{n,m,\lambda}$ is taken as the restriction of $\tilde{u}_{n,m,\lambda}$ to Ω .

A unique (continuous) solution is known to exist for any $\lambda > 0$ provided

$$m > d/2$$

$$n \geq M = \binom{m+d-1}{d}$$

and the "desion" $t_1, \dots, t_n$ is "unisolvant", that is, if $\{\phi_v\}_{v=1}^M$ are a basis for the M dimensional space of polynomials of total degree $m-1$ or less, then $\sum_{v=1}^M a_v \phi_v(t_i) = 0$, $i = 1, 2, \dots, n$, implies that the a_v are all 0. See Duchon (1976a,1976b), Meinquet (1978,1979), Paihua (1977,1978). We henceforth assume these conditions. Duchon has shown that the solution has a representation

$$\tilde{u}_{n,m,\lambda}(t) = \sum_{j=1}^n c_j E_m(t, t_j) + \sum_{v=1}^M d_v \phi_v(t),$$

where

$$E_m(s, t) = \theta_m |s-t|^{2m-d} \log |s-t| \quad m \text{ even} \\ = \theta_m |s-t|^{2m-d} \quad m \text{ odd}$$

$$\text{where, if } s = (x_1, \dots, x_d), t = (y_1, \dots, y_d), |s-t| = \left(\sum_{i=1}^d (x_i - y_i)^2 \right)^{1/2}, \text{ and} \\ \theta_m = (-1)^{d/2+1} / (2^{d/2-1} \pi^{d/2} (m-1)! (m-d/2)!) \quad m \text{ even} \\ = (-1)^m \Gamma(d/2-m) / 2^{2m} \pi^{d/2} (m-1)! \quad m \text{ odd}.$$

The coefficients $c = (c_1, \dots, c_n)'$ and $d = (d_1, \dots, d_M)'$ are determined by

$$(K + \rho I) c + T d = z \quad (1.2)$$

$$T' c = 0, \quad (1.3)$$

where K is the $n \times n$ matrix with jk th entry $E_m(t_j, t_k)$, $\rho = n\lambda$, T is the $n \times M$ matrix with i th entry $\phi_v(t_i)$ and $z = (z_1, \dots, z_n)'$. See Duchon (1976,1977), Paihua (1977,1978), Wahba (1979). We discuss the choice of λ shortly.

Let Ω be a closed, bounded subset in R^d . We will suppose that the $\{t_i\}$ become dense in Ω in such a way that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \rho(t_i) = \frac{1}{|\Omega|} \int_{\Omega} \rho(t) dt, \quad |\Omega| = \int_{\Omega} dt \quad (1.4)$$

for any continuous ρ . (However, it will be clear that our rate arguments hold under weaker conditions on the distribution of the $\{t_i\}$, for example

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \rho(t_i) = \int_{\Omega} \rho(t) w(t) dt$$

for some sufficiently nice positive w .) Let $R(\lambda)$ be the integrated mean square error when λ is used:

$$R(\lambda) = \frac{1}{n} \sum_{i=1}^n (u_{n,m,\lambda}(t_i) - u(t_i))^2 = \frac{1}{|\Omega|} \int_{\Omega} (u_{n,m,\lambda}(t) - u(t))^2 dt. \quad (1.5)$$

The smoothing parameter λ^* which minimizes $ER(\lambda)$ can be estimated by the

method of generalized cross-validation (GCV), see Craven and Wahba (1979), Golub,

Heath and Wahba (1977), Wahba (1979). Pleasing numerical results have been obtained

in Monte Carlo studies for $d=1$, $m=2$ (Craven and Wahba (1979)) and $d=2$, $m=2$, Wahba

(1979). Convergence rates for $ER(\lambda^*)$ have been obtained in the one dimensional case (Wahba (1975)).

Stone (1978) has recently obtained some rather general results on best

achievable pointwise convergence rates for the model (1.1), for any method of

estimation of $u(t)$. Reduced to our case and phrased loosely, his results say that

the rate

$$E(\hat{u}(t)-u(t))^2 = O(n^{-(2m-1)/(2m-1+d)}),$$

where $\hat{u}(t)$ is any estimate of $u(t)$ from the data $\underline{z}$, can be achieved for all $u \in H_m(\Omega)$ but not bettered. In this paper we are concerned with integrated mean square error

convergence rates: -

$$E \frac{1}{|\Omega|} \int_{\Omega} (u_{n,m,\lambda^*}(t) - u(t))^2 dt = ER(\lambda^*)$$

of u_{n,m,λ^*} .

It is our goal to give a plausible argument that

i) if $u \in H_m^m(\Omega)$,

$$ER(\lambda^*) = O(n^{-2m/(2m+d)})$$

and if

ii) $u \in H_{2m}^{2m}(\Omega)$ and some other conditions are satisfied, then

$$ER(\lambda^*) = O(n^{-4m/(4m+d)})$$

Our argument follows the arguments given in Wahba (1975, 1977) and Craven and Wahba (1979), and is given in section 2.

2. Plausibility arguments, numerical evidence

Let $A(\lambda)$ be the $n \times n$ matrix defined by

$$\begin{pmatrix} u_{n,m,\lambda}(t_1) \\ \vdots \\ u_{n,m,\lambda}(t_n) \end{pmatrix} = A(\lambda) \underline{z}.$$

If $R(\lambda)$ is taken as the middle quantity in (1.5), we have

$$R(\lambda) = \frac{1}{n} \|A(\lambda)(\underline{u} + \underline{\varepsilon}) - \underline{u}\|^2$$

where $\underline{u} = (u(t_1), \dots, u(t_n))'$, $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)'$, and

$$ER(\lambda) = \frac{1}{n} \| (I - A(\lambda)) \underline{u} \|^2 + \frac{\sigma^2}{n} \text{Trace } A^2(\lambda). \quad (2.1)$$

($A(\lambda)$ is symmetric.)

We call $\frac{1}{n} \| (I - A(\lambda)) \underline{u} \|^2$ the "squared bias" and $(\sigma^2/n) \text{Trace } A^2(\lambda)$ the variance.

Lemma 1.

$$\frac{1}{n} \| (I - A(\lambda)) \underline{u} \|^2 \leq \lambda J_m(\underline{u}) \quad (2.2)$$

where, for $\underline{v} \in \bar{X}$

$$J_m(\underline{v}) = \sum_{i=1}^d \frac{\partial^m \tilde{v}(x_1, \dots, x_d)^2}{\partial x_{i_1} \dots \partial x_{i_m}} d_{x_1} \dots d_{x_m}$$

and $\underline{u}$ is that element in $\bar{X}$ which minimizes J_m subject to coinciding with u on Ω .

Proof.

For each i , $\tilde{u}(t_i) = u(t_i)$. $A(\lambda)\underline{u}$ is a vector of values of the function, call it $\tilde{u}_{n,m,\lambda}^*$ which is the solution to the problem: Find $\tilde{v} \in \bar{X}$ to minimize

$$\frac{1}{n} \sum_{j=1}^n (u(t_j) - \tilde{v}(t_j))^2 + \lambda J_m(\tilde{v}).$$

Therefore

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n (u(t_j) - \tilde{u}_{n,m,\lambda}^*(t_j))^2 + \lambda J_m(\tilde{u}_{n,m,\lambda}^*) \\ &= \frac{1}{n} \| (I - A) \underline{u} \|^2 + \lambda J_m(\tilde{u}_{n,m,\lambda}^*) \\ &\leq \frac{1}{n} \sum_{j=1}^n (u(t_j) - \tilde{u}(t_j))^2 + \lambda J_m(\tilde{u}) = \lambda J_m(\underline{u}). \end{aligned}$$

We now investigate $\text{Trace } A^2(\lambda)$. Let $T_{n \times M}$ be the $n \times M$ dimensional matrix with j_v th entry $\phi_v(t_j)$. Let R be any $n \times (n-M)$ dimensional matrix of rank $n-M$ satisfying $R'T = 0_{(n-M) \times M}$, $R'R = I_{n-M}$. Following the results of Anselone and Laurent (1968) it is shown in Wahba (1979) that c and d satisfying (1.2) and (1.3) have the representations

$$\begin{aligned} c &= R(R'KR + pI)^{-1}R'z \\ d &= (T'T)^{-1}T'(z - Kc) \end{aligned}$$

and that

$$(I - A(\lambda))z = pc = n\lambda R(R'KR + n\lambda I)^{-1}R'z, \quad z \in E_n. \quad (2.3)$$

Hence, if we define $B = R'KR$ and let b_{vn} , $v = 1, 2, \dots, n-M$ be the $m-M$ eigenvalues of B , then

$$\frac{1}{n} \text{Tr } A^2(\lambda) = \frac{1}{n} \sum_{v=1}^n \left(\frac{b_{vn}}{b_{vn} + n\lambda} \right)^2 = \frac{1}{n} \sum_{v=1}^n \frac{1}{(1 + n\lambda/b_{vn})^2}.$$

We remark that K is not, in general, positive definite, however $R'KR$ is, since it is known that $r'Kr > 0$ for any non-trivial r satisfying $T'r = 0$ (See Páhuva (1977), Duchon (1977)).

Lemma 2.

Suppose there exist $p > 1$, and $k_1 > 0$ such that

$$\lim_{v \rightarrow \infty} \lim_{n \rightarrow \infty} \left| \frac{b_{vn}}{n} - \frac{k_1}{p} \right| \rightarrow 0$$

then, for some constant k_2 ,

$$\frac{1}{n} \text{Tr } A^2(\lambda) = \frac{k_2}{n\lambda} (1 + o(1)). \quad (2.4)$$

Outline of Proof.

$$\begin{aligned} \frac{1}{n} \text{Tr } A^2(\lambda) &= \frac{1}{n} \sum_{v=1}^n \frac{1}{(1 + n\lambda/b_{vn})^2} = \frac{1}{n} \sum_{v=1}^n \frac{1}{(1 + k_1^{-1} \lambda v p)^2} \\ &= \frac{1}{n} \int_0^\infty \frac{dx}{(1 + k_1^{-1} \lambda x p)^2} \\ &= \frac{k_2}{n\lambda^{1/p}} \end{aligned}$$

(A more rigorous argument can be found in Craven and Wahba 1979)).

Lemma 3. (Conjecture)

For $2m/d > 1$ there exists a constant k such that

$$\lim_{v \rightarrow \infty} \lim_{n \rightarrow \infty} \left| \frac{b_{vn}}{n} - \frac{k}{v^{2m/d}} \right| = 0. \quad (2.5)$$

Argument

We first argue that the eigenvalues $\lambda_1, \lambda_2, \dots$ of the integral operator K on $L_2(\Omega)$ defined by

$$(Ku)(t) = \int_{\Omega} E_m(t, s) u(s) ds$$

asymptotically behave like $\lambda_v = k/v^{2m/d}$, for some k , and then that this entails that the eigenvalues b_{vn} of K behave like $n\lambda_v/|\Omega|$, $v = 1, 2, \dots, n$, $n = 1, 2, \dots$.

Δ^m is a left inverse of K , since, if

$$\psi(t) = \int_{\Omega} E_m(t, s) \phi(s) ds$$

then $\Delta^m \psi(t) = \phi(t)$, $t \in \Omega$ (See Courant and Hilbert (1953)). Thus it is to be expected that the eigenvalues of K asymptotically decrease at the same rate as the eigenvalues of Δ^m increase. Let $d = 2$ and suppose Ω is the rectangle with sides a_1 and a_2 . The eigenfunctions $\{\phi\}$ and eigenvalues $\{\rho\}$ for the equation

$$\begin{aligned} \Delta u &= \rho u \\ \phi_{\xi, \eta}(x_1, x_2) &= \sin \frac{\xi \pi x_1}{a_1} \sin \frac{\eta \pi x_2}{a_2} \end{aligned}$$

$$\rho_{\xi, \eta} = \pi^2 \left(\frac{\xi^2}{a_1^2} + \frac{\eta^2}{a_2^2} \right), \quad \xi, \eta = 1, 2, \dots$$

It follows, by counting the number of pairs (ξ, η) in the ellipse $\pi^2 \left(\frac{\xi^2}{a_1^2} + \frac{\eta^2}{a_2^2} \right) \leq c$, that, if the eigenvalues $\rho_{\xi, \eta}$ ($\xi, \eta = 1, 2, \dots$) are reindexed in size place as ρ_v , $v = 1, 2, \dots$ that

$$\lim_{v \rightarrow \infty} \frac{\rho_v}{v} = \frac{4\pi}{|\Omega|}.$$

This relation is known to hold independently of the shape of Ω , and also for a Neumann boundary condition instead of $u = 0$ on $\partial\Omega$. Similarly eigenfunctions and eigenvalues for

$$\Delta^m u = \rho u$$

$$K^m u = 0 \text{ on } \partial\Omega, \quad k = 0, 1, \dots, m-1$$

are $\phi_{\varepsilon, n}$ and $\rho_{\varepsilon, n}^m$ so that the eigenvalues $\{\rho_v\}$ satisfy

$$\lim_{v \rightarrow \infty} \frac{\rho_v}{v^m} = \left(\frac{4\pi}{|\Omega|} \right)^m,$$

and this result is independent of the shape of Ω . Going to $d = 3$ dimensions, the eigenvalues for $\Delta u = \rho u$ on a rectangle with sides a_1, a_2 and a_3 and suitable boundary conditions are

$$\rho_{\varepsilon, n, \psi} = \pi^2 \left(\frac{\varepsilon^2}{a_1^2} + \frac{\psi^2}{a_2^2} + \frac{\psi^2}{a_3^2} \right) \quad \varepsilon, n, \psi = 1, 2, \dots$$

and, by counting the number of triplets (ε, n, ψ) in the ellipse

$$\pi^2 \left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \right) \leq c$$

one obtains that

$$\lim_{v \rightarrow \infty} \frac{\rho_v^{3/2}}{v} = \frac{6\pi^2}{|\Omega|}.$$

or

$$\frac{\rho_v}{v^{2/3}} = \left(\frac{6\pi^2}{|\Omega|} \right)^{2/3} (1 + o(1)).$$

See Courant and Hilbert (1953). Similarly the eigenvalues for Δ^m satisfy

$$\frac{\rho_v}{v^{2m/3}} = \left(\frac{6\pi^2}{|\Omega|} \right)^{2m/3} (1 + o(1))$$

and, extending the argument to d dimensions gives

$$\frac{\rho_v}{v^{2m/d}} = \left(\frac{(2\pi)^d}{V_d} \right)^{2m/d} (1 + o(1))$$

where V_d is the volume of the sphere of radius 1 in d dimensions. Therefore, we conjecture that the rate of decrease of the eigenvalues $\{\lambda_v\}$ of K is $v^{-2m/d}$.

Let $K(s, t)$ be a kernel with a Mercer-Hilbert Schmidt expansion on Ω ,

$$K(s, t) = \sum_{v=1}^{\infty} \lambda_v \phi_v(s) \phi_v(t)$$

where the eigenvalues $\{\lambda_v\}$ are absolutely summable and the eigenfunctions $\{\phi_v\}$ are an orthonormal set on $L_2(\Omega)$. Then, for large n ,

$$K(t_i, t_j) \approx \sum_{v=1}^n \lambda_v (\phi_v(t_i)/\sqrt{n}) (\phi_v(t_j)/\sqrt{n}),$$

and provided

$$\frac{1}{n} \sum_{i=1}^n \phi_v(t_i) \phi_\mu(t_i) = \int_{\Omega} \phi_v(t) \phi_\mu(t) dt = 1, \quad \mu = v = 0, \mu \neq v,$$

we see that eigenvalues λ_{v_n} , $v = 1, 2, \dots, n$, say, of the matrix K with j th entry

$K(t_j, t_k)$, have an approximation as $\lambda_{v_n} \sim n \lambda_v / |\Omega|$.

We have computed the eigenvalues b_{v_n} , $v = 1, 2, \dots, n-M$ for an example with $d = 2$, $m = 2$, and $n = 81$. Thus there are $n-M = 78$ eigenvalues. The t_i are arranged on a 9×9 square array. If $b_{v_n} \sim c v^{-p}$, then a plot of b_{v_n} vs. v on log-log paper should fall on a straight line with slope $-p$, here $p = 2$. Figure 1 gives a plot of these 78 eigenvalues. For comparison, a solid line has been drawn with slope -2 .

Theorem

Suppose Lemma 3 is true. Then, if $u \in H^m(\Omega)$,

$$\min_{\lambda} ER(\lambda) = O(n^{-2m/(2m+d)}).$$

Proof

By (2.1), (2.2), (2.4) and (2.5)

$$ER(\lambda) \leq c_1 \lambda + c_2 n \lambda^{d/2m}$$

where c_1 and c_2 are constants. Minimizing this expression with respect to λ gives $\lambda^* = O(n^{-2m/(2m+d)})$, where λ^* is the minimizer of $R(\lambda)$, and thence the result.

Lemma 4. (Conjecture)

Suppose u has a representation

$$u(t) = \int_{\Omega} E_m(t, s) \rho(s) ds + \sum_{v=1}^M \theta_v \phi_v(t) \quad (2.6)$$

where ρ is piecewise continuous and satisfies $\int_{\Omega} \phi_v(s) \rho(s) ds = 0$, $v = 1, 2, \dots, M$. Then

$$\frac{1}{n} \| (I - A(\lambda)) u \|^2 \leq \lambda^2 \int_{\Omega} |\phi_v|^2 dt (1 + o(1)). \quad (2.7)$$

Remark: If u has the given form, then $\rho = \Delta^m u$. However, the class of functions with representation (2.6) is restrictive since, for example it excludes harmonic functions other than polynomials of degree $\leq m$. (See Courant and Hilbert)

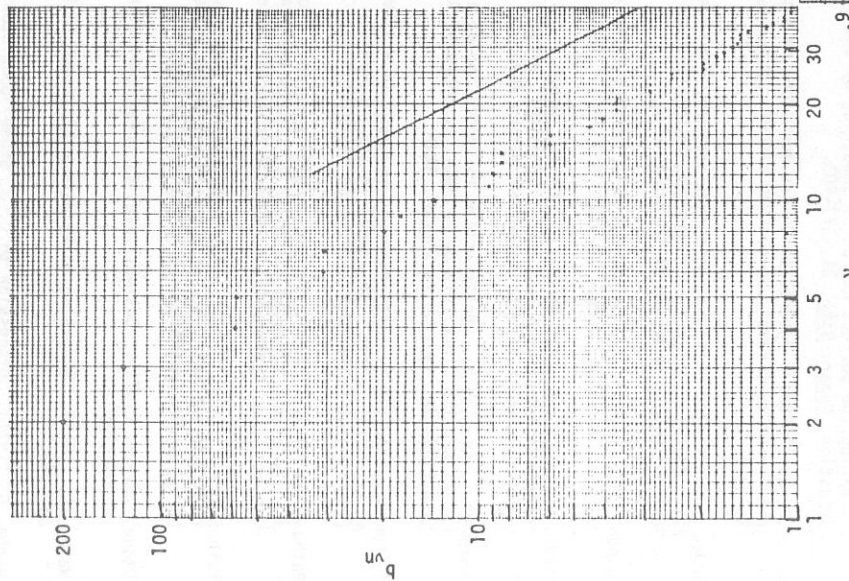
Suggestion of proof. From (2.3) we have

$$\frac{1}{n} \| (I - A(\lambda)) u \|^2 = n \lambda^2 u' R(RK' + n \lambda I)^{-2} R' u$$

and the right hand side is bounded above by

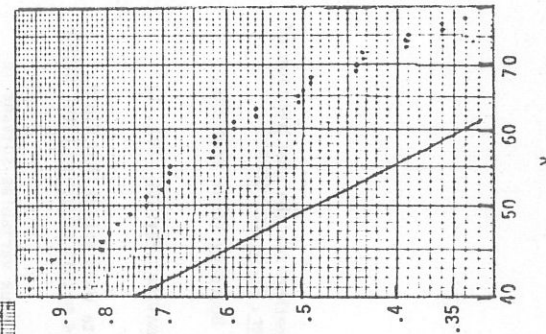
$$n \lambda^2 \| (RK R')^{-1} R' u \|^2$$

If u has the required form (2.6), then



Computed Eigenvalues, b_{vn} .

Figure 1



$$\tilde{u} = \frac{1}{n} K \tilde{\rho} + T \tilde{\theta} + \tilde{\delta}^1$$

where $\tilde{\rho} = (\rho(t_1), \dots, \rho(t_n))'$, $\tilde{\theta} = (\theta_1, \dots, \theta_M)'$ and $\tilde{\delta}^1 = (\delta_1^1, \dots, \delta_n^1)'$ is a vector of quadrature errors which we must assume are negligible in the limit. Similarly $T \tilde{\rho} = \tilde{\delta}^2$, where $\tilde{\delta}^2$ is a vector of quadrature errors which we must assume are negligible in the limit. Assuming $T \tilde{\rho} = 0$, then $\tilde{\rho} = R \tilde{\rho}_0$, for some n -M vector $\tilde{\rho}_0$.

Then

$$R' \tilde{u} = \frac{1}{n} R' K R \tilde{\rho}_0 + \text{negligible terms}$$

and

$$\begin{aligned} n \lambda^2 \tilde{u}' R (R K R' + n \lambda I)^{-2} R' \tilde{u} \\ \leq n \lambda^2 \tilde{u}' R (R K R')^{-2} R' \tilde{u} \\ = \frac{\lambda^2 |\tilde{\rho}_0|^2}{n} \|\tilde{\rho}_0\|^2 = \frac{\lambda^2 |\tilde{\rho}|^2}{n} \|\tilde{\rho}\|^2 = \lambda^2 |\tilde{\rho}|^2 \int_{\Omega} \rho^2(t) dt (1 + o(1)). \end{aligned}$$

Theorem 2.

Suppose

$$u(t) = \int_{\Omega} E_m(t, s) \rho(s) ds + \sum_{v=1}^M \theta_v \phi_v(t)$$

for some ρ piecewise continuous with $\int_{\Omega} \phi_v(s) \rho(s) ds = 0$, $v = 1, 2, \dots, M$. Then (assuming the conclusions of lemmas 3 and 4),

$$\min_{\lambda} R(\lambda) = O(n^{-4m/(4m+d)})$$

Proof.

Using (2.4), (2.5) and (2.7) gives

$$R(\lambda) \leq k_3 \lambda^2 + k_4 / n \lambda^{d/2m}$$

where k_3 and k_4 are constants. Setting $\lambda = O(n^{-2m/(4m+d)})$ gives the result.

References

- Anselone, P.M., and Laurent, P.J. (1968). A general method for the construction of interpolating or smoothing spline-functions, Numer. Math. 12, 66-82.
- Aubin, Jean-Pierre (1972). Approximation of Elliptic Boundary Value Problems, Wiley-Interscience.
- Courant, R. and Hilbert, D. (1953). Methods of Mathematical Physics, Vol. I, Interscience.
- Craven, P., and Wahba, G. (1979). Smoothing noisy data with spline functions: estimating the correct degree of smoothing by the method of generalized cross-validation, Numer. Math. 31, 377-403.
- Duchon, Jean (1976a). Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces. R.A.I.R.O. Analyse Numerique, 10, 12, pp.5-12.
- Duchon, Jean (1976b). Fonctions spline du type "Plaque mince" en dimension 2. No. 231. Seminaire d'analyse numerique, Mathematiques Appliquees, Universite Scientifique et Medicale de Grenoble.
- Duchon, Jean, (1977). Fonctions-spline a energie invariante par rotation, R.R.#27, Mathematiques Appliquees, Universite Scientifique et Medicale de Grenoble.
- Golub, G., Heath, M., and Wahba, G. (1977). Generalized cross-validation as a method for choosing a good ridge parameter. University of Wisconsin-Madison, Department of Statistics Technical Report #491, to appear, Technometrics.
- Meinguet, Jean, (1978). Multivariate interpolation at arbitrary points made simple. Report No. 118, Institut de Mathematique Pure et Appliquee, Universite Catholique de Louvain, to appear, ZAMP.
- Meinguet, Jean, (1979). An intrinsic approach to multivariate spline interpolation at arbitrary points. To appear in the Proceedings of the NATO Advanced Study Institute on Polynomial and Spline Approximation, Calgary 1978, B. Sahney, Ed.
- Paihua, Montes, L. (1977). Methodes numeriques pour l'obtention de fonctions-spline du type plaque mince en dimension 2. Seminaire d'Analyse Numerique No. 273, Mathematiques Appliquees, Universite Scientifique et Medicale de Grenoble.
- Paihua Montes, Luis, (1978). Quelques methodes numeriques pour le calcul de fonctions splines a une et plusieurs variables. Thesis, Analyse Numerique Universite Scientifique et Medicale de Grenoble.
- Stone, Charles (1978). Optimal rates of convergence for non-parametric estimators (manuscript).
- Wahba, G. (1975). Smoothing noisy data with spline functions, Numer. Math., 24, 383-393.
- Wahba, G. (1977). Discussion to C. J. Stone Consistent nonparametric regression, Ann. Statist. 5, 637-640.
- Wahba, Grace (1979). How to smooth curves and surfaces with splines and cross-validation. University of Wisconsin-Madison, Department of Statistics TR#555.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)		REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER		4. TYPE OF REPORT & PERIOD COVERED	
TR#557				5. PERFORMING ORG. REPORT NUMBER	
4. TITLE (and Subtitle) CONVERGENCE RATES OF "THIN PLATE" SMOOTHING SPLINES WHEN THE DATA ARE NOISY		6. CONTRACT OR GRANT NUMBER(s)		7. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
7. AUTHOR(s) Grace Wahba		8. CONTRACT OR GRANT NUMBER(s) ONR # N00014-77-C-0675		9. REPORT DATE March 1979	
10. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 N. Quincy Street Arlington, VA 22217		11. NUMBER OF PAGES 13		12. SECURITY CLASS. (of this report) Unclassified	
13. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		14. DISTRIBUTION STATEMENT (of this Report) Distribution of this document is unlimited		15. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of the abstract entered in Block 20. If different from Report)					
17. SUPPLEMENTARY NOTES					
18. KEY WORDS (Continue on reverse side if necessary and identify by block number) multidimensional spline smoothing generalized cross validation surface smoothing					
19. ABSTRACT (Continue on reverse side if necessary and identify by block number) We study the use of "thin plate" smoothing splines for smoothing noisy d dimensional data. The model is $z_i = u(t_i) + \epsilon_i, \quad i = 1, 2, \dots, n,$ where u is a real valued function on a closed, bounded subset Ω of Euclidean d -space and the ϵ_i are random variables satisfying $E\epsilon_i = 0$, $E\epsilon_i \epsilon_j = \sigma^2 \delta_{ij}$, $i, j \in \Omega$. The z_i are observed. It is desired to estimate u , given $z_1, \dots, z_n$. u is only assumed to be "smooth", more precisely we assume that u is in the Sobolev space $H^s(\Omega)$.					

space $H^m(\Omega)$ of functions with partial derivatives up to order m in $L_2(\Omega)$, with $m > d/2$. u is estimated by $u_{n,m,\lambda}$, the restriction to Ω of $\tilde{u}_{n,m,\lambda}$, where $\tilde{u}_{n,m,\lambda}$ is the solution to: Find $\tilde{u}$ (in an appropriate space of functions on R^d) to minimize

$$\frac{1}{n} \sum_{i=1}^n (\tilde{u}(t_i) - z_i) + \lambda \sum_{i_1, \dots, i_m=1}^d \int_{R^d} \left(\frac{\partial^m \tilde{u}}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_m}} \right)^2 dx_1, dx_2, \dots, dx_d.$$

This minimization problem is known to have a solution for $\lambda > 0$, $m > \frac{d}{2}$, $n \geq M \binom{m+d-1}{d}$, provided the $t_1, \dots, t_n$ are "unisolvent". We consider the integrated mean square error

$$R(\lambda) = \frac{1}{|\Omega|} \int_{\Omega} (u_{n,m,\lambda}(t) - u(t))^2 dt, \quad |\Omega| = \int_{\Omega} dt,$$

and $ER(\lambda)$, as $\{t_i\}_{i=1}^n$ become dense in Ω . An estimate of λ which asymptotically minimizes $ER(\lambda)$ can be obtained by the method of generalized cross-validation. In this paper we give plausible arguments and numerical evidence supporting the following conjectures:

Suppose $u \in H^m(\Omega)$. Then

$$\min_{\lambda} ER(\lambda) = O(n^{-2m/(2m+d)}).$$

Suppose $u \in H^{2m}(\Omega)$ and certain other conditions are satisfied. Then

$$\min_{\lambda} ER(\lambda) = O(n^{-4m/(4m+d)}).$$