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ESTIMATION OF THE COEFFICIENTS IN A MULTIDIMENSIONAL DISTRIBUTED LAG MODEL[1]

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Least squares type estimates of the coefficients $b(\tau)$, $\tau = 0, \pm 1, \pm 2, \ldots$, in the general multidimensional distributed lag model

$$Y(t) = \sum_{s=-\infty}^{\infty} b(s)X(t-s) + \varepsilon(t) \qquad (t = \ldots -1, 0, 1, \ldots)$$

are considered, where $\{X(t)\}$ and $\{Y(t)\}$ are observable random processes, $\varepsilon(t)$ is an unobservable noise process. The asymptotic joint distribution of the estimates, conditional on the observed spectral density of the input $X(t)$, is given, as well as the unconditional first and second moments, a readily computable confidence ellipsoid, and an approximate expression for the expected covariance in predicting $Y(t)$ from a new realization of $X(t)$, using the estimated coefficients.

1. INTRODUCTION

LET

$$(1.1) \qquad Y(t) = \sum_{s=-\infty}^{\infty} b(s)X(t-s) + \varepsilon(t) \qquad (t = \ldots -1, 0, 1, 2, \ldots),$$

where $X(s)$ (column vector) is a P-dimensional covariance stationary random process, $b(\tau)$ is a $Q \times P$ matrix for each τ, with

$$b(\tau) = \left\{ b_{jk}(\tau), \sum_{j=1}^{Q} \sum_{k=1}^{P} \sum_{s=-\infty}^{\infty} |b_{jk}(s)| < \infty \right\},$$

and $Y(t)$ and $\varepsilon(t)$ are Q-dimensional covariance stationary random processes with the series $\{\varepsilon(t)\}$ independent of the series $\{X(t)\}$. Sample functions of $X(t)$ and $Y(t)$ are observed for $t = 1, 2, \ldots, T$, and it is desired to estimate $b(\tau)$ for some of the τ. In this note we give the asymptotic joint distribution of certain least squares estimates $\hat{b}(\tau)$ of $b(\tau)$, under the assumption that the time series are Gaussian. A readily computable confidence ellipsoid for the estimates is given, as well as an approximate expression for the expected covariance in predicting $Y(t)$ from a new realization of $X(t)$, using the estimated coefficients. The results allow a test of the hypothesis $b(\tau) \approx b^0(\tau)$. Intermediate results can be used to provide a test of the hypothesis $B(\omega) \approx B^0(\omega)$, where $B(\omega) = \sum_{\tau=-\infty}^{\infty} b(\tau)\, e^{i\omega\tau}$ is the transfer characteristic in the frequency domain.

Estimates of the type considered here were introduced by Hannan [**3**] for the case $P = Q = 1$. They are formed from estimates of the auto and cross spectral densities of the series $X(t)$ and $Y(t)$. In a recent paper Hannan [**5**] obtains a central limit theorem which gives the asymptotic covariance of the estimates for general P and Q. His method of proof is different from that presented here. He does not

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assume normality of the $X(t)$ and $Y(t)$ series, and considers a general class of windowed estimates for the auto and cross spectral densities where the windows satisfy some conditions. He assumes that the entries of $b(\tau)$ are 0 for all (τ) greater than some q. (See [**5**] for details.) In this note the effects of $b(\tau) \neq 0$ for large τ are exhibited, and furthermore the results hold for estimates of $b(\tau)$ for large τ provided the length of the record is also long. Let M be the bandwidth parameter of the windows used for estimation of spectral densities. Loosely speaking the window is of width $2\pi/M$. Hannan requires for his results that $M^2/T \to 0$, $T/M^{4t+4} \to 0$, where T is the length of the record, and t is a nonnegative constant depending on the window type and the parameters of $Y(t)$. We consider only a special type of estimate of the spectral quantities, that is, averages of appropriately spaced periodograms.[2] In return, the asymptotic formulas hold provided only that $(M \log M)/T \to 0$, $1/M \to 0$.

2. THE ESTIMATES

Let the auto and cross covariance (matrix) functions be defined by

$$\begin{aligned} R^{XX}(\tau) &= EX(t)X'(t+\tau), \\ R^{XY}(\tau) &= EX(t)Y'(t+\tau), \qquad (2.1) \\ R^{YY}(\tau) &= EY(t)Y'(t+\tau). \end{aligned}$$

We assume that the joint spectral density matrix,

$$F(\omega) = \begin{pmatrix} F^{XX}(\omega) & F^{XY}(\omega) \\ F^{YX}(\omega) & F^{YY}(\omega) \end{pmatrix}, \qquad (2.2)$$

of the X and Y processes exists and is strictly positive definite, where

$$\begin{aligned} F^{XX}(\omega) &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-i\omega\tau} R^{XX}(\tau), \\ F^{XY}(\omega) &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-i\omega\tau} R^{XY}(\tau), \qquad (2.3) \\ F^{YY}(\omega) &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-i\omega\tau} R^{YY}(\tau). \end{aligned}$$

Letting

$$B(\omega) = \sum_{\tau=-\infty}^{\infty} b(\tau)\, e^{i\omega\tau}, \qquad (2.4)$$

it is well known that

$$B(\omega) = (F^{XX}(\omega))^{-1} F^{XY}(\omega), \qquad (2.5)$$

[2] The periodograms are averaged within a frequency band of width $2\pi/M$.

and that

$$b^M(s) \stackrel{\text{def}}{\equiv} \sum_{\nu=-\infty}^{\infty} b(s+\nu M) = \frac{1}{M}\sum_{\nu=1}^{M} B\left(\frac{2\pi\nu}{M}\right) e^{-2\pi i\nu s/M} \quad (s = 0, \pm 1, \pm 2, \ldots). \tag{2.6}$$

Since, with a finite record, one cannot estimate an infinite sequence of parameters, we instead choose an M and estimate the "aliased" version $b^M(s)$ of $b(s)$ for $|s| \leqslant (M-1)/2$. If $b(s) \equiv 0$ for $|s| > (M-1)/2$, then $b^M(s) \equiv b(s)$ for $|s| \leqslant (M-1)/2$.

Let

$$\hat{F}(\omega) = \begin{pmatrix} \hat{F}^{XX}(\omega) & \hat{F}^{XY}(\omega) \\ \hat{F}^{YX}(\omega) & \hat{F}^{YY}(\omega) \end{pmatrix} \quad \text{for} \quad \omega = \frac{2\pi\nu}{M} \qquad (\nu = 1, 2, \ldots, M), \tag{2.7}$$

where the elements $\hat{f}(\omega)$ of $\hat{F}(\omega)$ are estimates of the indicated spectral quantities, to be identified below, estimated from records of length T.

Estimates $\hat{b}(s)$ for $b^M(s)$ suggested by (2.6) are

$$\hat{b}(s) = \frac{1}{M}\sum_{\nu=1}^{M} \hat{B}\left(\frac{2\pi\nu}{M}\right) e^{-2\pi i\nu s/M}, \quad |s| \leqslant \frac{M-1}{2}, \tag{2.8}$$

where, for convenience, we assume M odd and

$$\hat{B}(\omega) = [\hat{F}^{XX}(\omega)]^{-1}\hat{F}^{XY}(\omega). \tag{2.9}$$

The reason for calling these estimates least squares estimates is discussed in Section 6.

3. THE ASYMPTOTIC JOINT DISTRIBUTION OF THE ESTIMATES

We now assume

PROPOSITION A: *The random matrices $\hat{F}(2\pi\nu/M)$, $\nu = 1, 2, \ldots, M$, satisfy $\hat{F}(2\pi(M-\nu)/M) \equiv \hat{F}^*(2\pi\nu/M)$ where "*" is complex conjugate, and are otherwise independent, and each has the complex Wishart distribution*[3] *$W_c(F(2\pi\nu/M), P+Q, n)$,* [**2**, **6**], *with n degrees of freedom, where $n \approx T/2M$, T, M and n are large.*

Proposition A was first suggested by Goodman in 1957 [**1**]. Define

$$\hat{F}^{XX}\left(\frac{2\pi\nu}{M}\right) = \frac{1}{M}\sum_{j=(\nu/M)-((n-1)/2)}^{(\nu/M)+((n-1)/2)} \frac{1}{2\pi T}\left(\sum_{r=1}^{T} X(r)\, e^{2\pi i r j/T}\right)\left(\sum_{s=1}^{T} X(s)\, e^{-2\pi i s j/T}\right), \tag{3.1a}$$

$$n \approx \frac{T}{2M} \qquad (\nu = 1, 2, \ldots, M).$$

Similarly define $\hat{F}^{XY}(2\pi\nu/M)$ and $\hat{F}^{YY}(2\pi\nu/M)$. Let $\hat{F}(\omega)$ of (2.7) be given by

$$\hat{F}\left(\frac{2\pi\nu}{M}\right) = \begin{pmatrix} \hat{F}^{XX}\left(\frac{2\pi\nu}{M}\right) & \hat{F}^{XY}\left(\frac{2\pi\nu}{M}\right) \\ \hat{F}^{YX}\left(\frac{2\pi\nu}{M}\right) & \hat{F}^{YY}\left(\frac{2\pi\nu}{M}\right) \end{pmatrix} \qquad (\nu = 1, 2, \ldots, M). \tag{3.1b}$$

[3] The complex Wishart distribution is a direct complex analogue of the real Wishart distribution. See (3.5) and the remarks following it for the complex Wishart density.

That is, the entries of the sample spectral density matrices are formed from non-overlapping averages of n neighboring periodograms.

The assumption of Proposition A is justified as follows. Using a theorem proved in Wahba [**8**] and quoted in the Appendix, it is shown in the Appendix that if $\hat{F}(\omega)$, $\omega = 2\pi\nu/M$, $\nu = 1, 2, \ldots, M$, are estimated as in (3.1), and if $\hat{b}(s), |s| \leqslant (M-1)/2$, are estimated as in (2.8), then there exists

$$\tilde{F}\left(\frac{2\pi\nu}{M}\right) = \begin{pmatrix} \tilde{F}^{XX}\left(\frac{2\pi\nu}{M}\right) & \tilde{F}^{XY}\left(\frac{2\pi\nu}{M}\right) \\ \tilde{F}^{YX}\left(\frac{2\pi\nu}{M}\right) & \tilde{F}^{YY}\left(\frac{2\pi\nu}{M}\right) \end{pmatrix} \qquad (\nu = 1, 2, \ldots, M),$$

a suitably constructed family of random matrices satisfying Proposition A, such that if

$$\tilde{b}(s) = \frac{1}{M}\sum_{\nu=1}^{M} \tilde{B}\left(\frac{2\pi\nu}{M}\right) e^{-2\pi i\nu s/M}, \quad |s| \leqslant \frac{M-1}{2}, \tag{3.2}$$

$$\tilde{B}(\omega) = [\tilde{F}^{XX}(\omega)]^{-1}\tilde{F}^{XY}(\omega),$$

then

$$E\frac{|\hat{b}_{\mu\nu}(s) - \tilde{b}_{\mu\nu}(s)|}{\sigma(\tilde{b}_{\mu\nu}(s))} \leqslant \text{const}\left[\frac{M\log M}{T} + \frac{1}{M}\right]^{\frac{1}{2}} \to 0 \quad \text{as} \quad M \to \infty,$$

$$\frac{M\log M}{T} \to 0 \qquad (\mu = 1, 2, \ldots, P;\ \nu = 1, 2, \ldots, Q),$$

where $\hat{b}_{\mu\nu}(s)$, $\tilde{b}_{\mu\nu}(s)$ are the μ, νth entries of $\hat{b}(s)$, $\tilde{b}(s)$ respectively, and $\sigma(\tilde{b}_{\mu\nu}(s))$ is the standard deviation of $\tilde{b}_{\mu\nu}(s)$. This result, of course entails convergence in distribution. The proof assumes that $F(\omega)$ is strictly positive definite, all ω, with $\sum_{\tau=-\infty}^{\infty} |\tau||R(\tau)| \leqslant \theta < \infty$, where $R(\tau)$ stands for the ξ, ηth entry of $R^{XX}(\tau)$, $R^{YY}(\tau)$ or $R^{XY}(\tau)$.

THEOREM: *Let Proposition A be satisfied and let $\hat{b}(s)$, $|s| \leqslant (M-1)/2$ be given by* (2.8). *Then*:

(*i*) *the conditional distribution of $\hat{b}(s)$, $|s| \leqslant (M-1)/2$, given $\hat{F}^{XX}(2\pi\nu/M)$, $\nu = 1, 2, \ldots, M$, is multivariate normal with*

$$E[\hat{b}_{\mu\nu}(s)] = b^{M}_{\mu\nu}(s), \quad \textit{and}$$

$$cov(\hat{b}_{\mu\nu}(s), \hat{b}_{\xi\eta}(t)|\hat{F}^{XX}) = \frac{1}{n}\frac{1}{M^2}\sum_{j=1}^{M} f^{ee}_{\mu\xi}\left(\frac{2\pi j}{M}\right)\hat{f}^{\nu\eta}_{XX}\left(\frac{2\pi j}{M}\right) e^{-(2\pi i j(s-t))/M}, \tag{3.3}$$

$$(\mu, \xi = 1, 2, \ldots, Q;\ \nu, \eta = 1, 2, \ldots, P),$$

where

$$\hat{b}(\tau) = \{\hat{b}_{\mu\nu}(\tau)\},$$

$$F^{ee}(\omega) = F^{YY}(\omega) - F^{YX}(\omega)(F^{XX}(\omega))^{-1}F^{XY}(\omega) = \{f^{ee}_{\mu\xi}\},$$

$$(\hat{F}^{XX}(\omega))^{-1} = \{\hat{f}^{\nu\eta}_{XX}(\omega)\}.$$

(*ii*) *The unconditional mean and covariance of* $\hat{b}(s)$ *is*

$$E\hat{b}_{\mu\nu}(s) = b^M_{\mu\nu}(s), \tag{3.4}$$
$$cov(\hat{b}_{\mu\nu}(s), \hat{b}_{\xi\eta}(t)) = \frac{1}{n-P}\frac{1}{M^2}\sum_{j=1}^{M} f^{ee}_{\mu\xi}\left(\frac{2\pi j}{M}\right) f^{\nu\eta}_{XX}\left(\frac{2\pi j}{M}\right) e^{-(2\pi i j(s-t))/M},$$

where

$$(F^{XX}(\omega))^{-1} = \{f^{\nu\eta}_{XX}(\omega)\}.$$

PROOF: Let

$$\hat{F}^{ee}\left(\frac{2\pi\nu}{M}\right) = \hat{F}^{YY}\left(\frac{2\pi\nu}{M}\right) - \hat{F}^{YX}\left(\frac{2\pi\nu}{M}\right)\left(\hat{F}^{XX}\left(\frac{2\pi\nu}{M}\right)\right)^{-1}\hat{F}^{XY}\left(\frac{2\pi\nu}{M}\right).$$

In what follows the argument $2\pi\nu/M$ is suppressed. Part (i) follows immediately from an expression given by Khatri [**6**] for the joint density of $\hat{F}^{ee}$, $\hat{B}$, $\hat{F}^{XX}$, when $\hat{F}$ is complex Wishart with n degrees of freedom.

It is

$$\begin{aligned} g(n\hat{F}^{ee}, \hat{B}, n\hat{F}^{XX}) = {} & \frac{1}{\Gamma_Q(n-P)} \frac{|n\hat{F}^{ee}|^{n-P-Q}}{|F^{ee}|^{n-P}} e^{-\operatorname{tr} F^{ee^{-1}}(n\hat{F}^{ee})} \\ & \times \frac{1}{\pi^{PQ}} \frac{|n\hat{F}^{XX}|^{Q}}{|F^{ee}|^{P}} e^{-\operatorname{tr} F^{ee^{-1}}(\hat{B}-B)n\hat{F}^{XX}(\hat{B}-B)^{*\prime}} \\ & \times \frac{1}{\Gamma_P(n)} \frac{|n\hat{F}^{XX}|^{n-P}}{|F^{XX}|^{n}} e^{-\operatorname{tr} F^{XX^{-1}}(n\hat{F}^{XX})}, \end{aligned} \tag{3.5}$$

where $\Gamma_r(n) = \pi^{\frac{1}{2}r(r-1)}\Gamma(n)\Gamma(n-1)\ldots\Gamma(n-r+1)$.

It is observed from the form of (3.5) that $\hat{F}^{ee}$ is distributed independently of $\hat{B}$ and $\hat{F}^{XX}$. Furthermore, $\hat{F}^{XX}$, being a submatrix on the diagonal of a complex Wishart distributed matrix, is also complex Wishart distributed, $W_c(F^{XX}, P, n)$. The third factor in (3.5) is the density function $W_c(F^{XX}, P, n)$, the distribution of $\hat{F}^{XX}$. Hence, the second factor in (3.5) is the conditional distribution of $\hat{B}$, given $\hat{F}^{XX}$. The conditional distribution of the entries $\hat{B}_{\mu\nu}$, $\mu = 1, 2, \ldots, Q$, $\nu = 1, 2, \ldots, P$, of $\hat{B}$, given $\hat{F}^{XX}$, are seen from the second factor of (3.5) to have the complex multivariate normal distribution [see **2**] with

$$E[\hat{B}_{\mu\nu}|\hat{F}^{XX}] = B_{\mu\nu}; \tag{3.6}$$

$$\operatorname{cov}(\hat{B}_{\mu\nu}, \hat{B}^*_{\xi\eta}|\hat{F}^{XX}) = \frac{1}{n} f^{ee}_{\mu\xi}\hat{f}^{\nu\eta}_{XX}. \tag{3.7}$$

Part (i) then follows from (2.6), (2.8), and the independence of the spectral estimates at distinct frequencies. Part (ii) follows from the fact that $E(\hat{F}^{XX})^{-1} = (n/(n-P))(F^{XX})^{-1}$. (This fact is proved in [7]. Its real analogue seems to be common knowledge.)

4. CONFIDENCE ELLIPSOIDS

A joint confidence ellipsoid for $\hat{b}(s)$, $|s| \leqslant (M-1)/2$ may be established assuming Proposition A, and the observation in the proof of the theorem. We indicate the details for the case $P = 2$, $Q = 1$.

Let b_j and $\hat{b}_j$ be the M-dimensional vectors

$$\left.\begin{aligned} b_j &= \left(b_j\left(-\frac{M-1}{2}\right), \ldots, b_j(v), \ldots, b_j\left(\frac{M-1}{2}\right)\right) \\ \hat{b}_j &= \left(\hat{b}_j\left(-\frac{M-1}{2}\right), \ldots, \hat{b}_j(v), \ldots, \hat{b}_j\left(\frac{M-1}{2}\right)\right) \end{aligned}\right\} \qquad (j = 1, 2),$$

where $b_j(\tau)$ and $\hat{b}_j(\tau)$ are the jth entries in the 1×2 matrices $b^M(\tau)$ and $\hat{b}(\tau)$.

Let b and $\hat{b}$ be the $2M$-dimensional vectors,

$$b = (b_1 : b_2),$$

$$\hat{b} = (\hat{b}_1 : \hat{b}_2).$$

Using (3.3), the $2M \times 2M$ conditional covariance matrix of $\hat{b}$, conditioned on $\hat{F}^{XX}(2\pi v/M)$, is given by Σ where

$$\Sigma = \frac{1}{nM}\begin{pmatrix} WD^{11}W^* & WD^{12}W^* \\ WD^{21}W^* & WD^{22}W^* \end{pmatrix}, \tag{4.1}$$

where W is the $M \times M$ unitary matrix with r, sth entry $(1/\sqrt{M})\, e^{-2\pi i r s/M}$, $D^{v\eta}$ is the $M \times M$ diagonal matrix with r, rth entry $f^{ee}(2\pi r/M)\hat{f}^{v\eta}_{XX}(2\pi r/M)$, $v, \eta = 1, 2$.

Let

$$\hat{\Sigma}^{-1} = nM\begin{pmatrix} W\hat{D}_{11}W^* & W\hat{D}_{12}W^* \\ W\hat{D}_{21}W^* & W\hat{D}_{22}W^* \end{pmatrix}, \tag{4.2}$$

where $\hat{D}_{v\eta}$ is the diagonal matrix with r, rth entry $\hat{f}^{XX}_{v\eta}(2\pi r/M)/\hat{f}^{ee}(2\pi r/M)$. We have the following lemma.

LEMMA:

$$(\hat{b} - b)\hat{\Sigma}^{-1}(\hat{b} - b)' \sim 2\sum_{s=1}^{M} t_s, \tag{4.3}$$

where $t_s = t_{M-s}$ and are otherwise independent F random variables with 4 and $2n$ degrees of freedom.

PROOF: A direct calculation shows that

$$(\hat{b} - b)\hat{\Sigma}^{-1}(\hat{b} - b)' = \sum_{s=1}^{M} \frac{f^{ee}\left(\frac{2\pi s}{M}\right)}{\hat{f}^{ee}\left(\frac{2\pi s}{M}\right)} \tag{4.4}$$

$$\times \left\{ n\left(\hat{B}\left(\frac{2\pi s}{M}\right) - B\left(\frac{2\pi s}{M}\right)\right) S\left(\frac{2\pi s}{M}\right)\left(\hat{B}\left(\frac{2\pi s}{M}\right) - B\left(\frac{2\pi s}{M}\right)\right)^{*\prime} \right\},$$

where

$$B\left(\frac{2\pi s}{M}\right) = \left(B_1\left(\frac{2\pi s}{M}\right), B_2\left(\frac{2\pi s}{M}\right)\right),$$

$$\hat{B}\left(\frac{2\pi s}{M}\right) = \left(\hat{B}_1\left(\frac{2\pi s}{M}\right), \hat{B}_2\left(\frac{2\pi s}{M}\right)\right),$$

$$B_j\left(\frac{2\pi s}{M}\right) = \sum_{v=-(M-1)/2}^{(M-1)/2} b_j(v)\, e^{2\pi i s v/M} \qquad (j = 1, 2),$$

$$\hat{B}_j\left(\frac{2\pi s}{M}\right) = \sum_{v=-(M-1)/2}^{(M-1)/2} \hat{b}_j(v)\, e^{2\pi i s v/M},$$

and

$$S\left(\frac{2\pi v}{M}\right) = \frac{1}{f^{ee}\left(\frac{2\pi v}{M}\right)} \hat{F}^{XX}\left(\frac{2\pi v}{M}\right).$$

Since the conditional distribution of $(\hat{B}(2\pi s/M) - B(2\pi s/M))$, conditioned on $\hat{F}^{XX}(2\pi v/M)$, is complex normal with (complex) covariance $\{nS(2\pi v/M)\}^{-1}$, the quantity in brackets in (4.4) is distributed as 1/2 times chi-square with $2P = 4$ degrees of freedom, independently of $\hat{f}^{ee}(2\pi s/M)/f^{ee}(2\pi s/M)$, which is distributed as $1/2n$ times chi-square with $2n$ degrees of freedom.

5. PREDICTION ERROR USING ESTIMATED COEFFICIENTS

Suppose the model represents some stationary physical process; one observes realizations $X(t)$ and $Y(t)$ and estimates $b(s)$. Later one observes, say, a new realization $\tilde{X}(t)$ of the same process and wishes to predict $\tilde{Y}(t)$ based on $\tilde{X}(t)$ and $\hat{b}(s)$. To avoid uninteresting complications, we assume that $|b(s)| = 0$ for $|s| > (M - 1)/2$.

Let the prediction $\hat{Y}(t)$ of $\tilde{Y}(t)$ be given by

$$\hat{Y}(t) = \sum_{|s| \leq (M-1)/2} \hat{b}(s)\tilde{X}(t - s)$$

where

$$\tilde{Y}(t) = \sum_{|s| \leq (M-1)/2} b(s)\tilde{X}(t - s) + \tilde{\varepsilon}(t)$$

and $\tilde{X}(t)$ and $\tilde{Y}(t)$ are new realizations of the $\{X(t), Y(t)\}$ processes.

We have the following corollary.

COROLLARY:

$$(5.1) \qquad E(\hat{Y}(t) - \tilde{Y}(t))(\hat{Y}(t) - \tilde{Y}(t))' \approx \left(1 + \frac{P}{n - P}\right) E\tilde{\varepsilon}(t)\tilde{\varepsilon}'(t),$$

for M large.

PROOF:

$$
\begin{aligned}
&E(\hat{Y}(t) - \tilde{Y}(t))(\hat{Y}(t) - \tilde{Y}(t))' \\
&\quad = E\left\{ \sum_{|r| \leqslant (M-1)/2} [\hat{b}(r) - b(r)]\tilde{X}(t-r) - \tilde{\varepsilon}(t) \right\} \\
&\qquad \times \left\{ \sum_{|s| \leqslant (M-1)/2} [\hat{b}(s) - b(s)]\tilde{X}(t-s) - \tilde{\varepsilon}(t) \right\}' \\
&\quad = \sum_{|r|,|s| \leqslant (M-1)/2} E[\hat{b}(r) - b(r)]R^{XX}(r-s)[\hat{b}(s) - b(s)]' \\
&\qquad + E\tilde{\varepsilon}(t)\tilde{\varepsilon}'(t),
\end{aligned} \tag{5.2}
$$

by the presumed independence of $\hat{b}$, $\tilde{X}$ and $\tilde{\varepsilon}$.

Now, the μ, ξth element of the matrix sum in the last line of (5.2) is given, using (3.4), by

$$
\begin{aligned}
&\frac{1}{n-P}\frac{1}{M}\sum_{j=1}^{M} f_{\mu\xi}^{ee}\left(\frac{2\pi j}{M}\right) \sum_{\nu,\eta=1}^{P} f_{XX}^{\nu\eta}\left(\frac{2\pi j}{M}\right) \\
&\quad \times \frac{1}{M}\sum_{|r|,|s| \leqslant (M-1)/2} R_{\nu\eta}(r-s)\, e^{-(2\pi i j(r-s))/M}.
\end{aligned} \tag{5.3}
$$

But, for M large,

$$
\begin{aligned}
&\frac{1}{M}\sum_{|r|,|s| \leqslant (M-1)/2} R_{\nu\eta}(r-s)\, e^{-(2\pi i j(r-s))/M} \\
&\quad = \sum_{\tau=-(M-1)/2}^{(M-1)/2} \left(1 - \frac{|\tau|}{M}\right) R_{\nu\eta}(\tau)\, e^{-2\pi i\tau/M} \approx 2\pi f_{\nu\eta}^{XX}\left(\frac{2\pi j}{M}\right),
\end{aligned} \tag{5.4}
$$

$$
\sum_{\nu,\eta=1}^{P} f_{XX}^{\nu\eta}\left(\frac{2\pi j}{M}\right) f_{\nu\eta}^{XX}\left(\frac{2\pi j}{M}\right) = P, \tag{5.5}
$$

$$
\left\{\frac{2\pi}{M}\sum_{j=1}^{M} f_{\mu\xi}^{ee}\left(\frac{2\pi j}{M}\right)\right\} \approx E\tilde{\varepsilon}_{\eta}(t)\tilde{\varepsilon}_{\xi}(t), \tag{5.6}
$$

giving the result.

6. DISCUSSION

The efficiency of estimates of this type is discussed by Hannan [**4**, **5**].

If we consider the observed values of the process $X(t)$ as fixed constants, then the estimates here are approximately the ordinary least squares estimates. We illustrate this remark with $P = Q = 1$. Fix $M \ll T$, M odd, and consider the model

$$
Y(t) = \sum_{|s| \leqslant (M-1)/2} b(s)X(t-s) + \varepsilon(t). \tag{6.1}
$$

If we know $X(t)$ for $1 - ((M-1)/2) \leqslant t \leqslant T + ((M-1)/2)$ and observe $Y(t)$ for $1 \leqslant t \leqslant T$, we may rewrite (6.1) as

$$
Y = \Xi b + \varepsilon \tag{6.2}
$$

where $Y = ((Y(1), Y(2), \ldots Y(T))', b = (b(-(M-1)/2), \ldots, b(0), \ldots, b((M-1)/2)'$, $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(T))'$ and Ξ is the $T \times M$ matrix with $X(j-k+((M-1)/2)+1)$ in the j, kth place, $j = 1, 2, \ldots, T$, $k = 1, 2, \ldots, M$. Hence $\hat{b}_{LS}$, the least squares estimate of b, is

$$\hat{b}_{LS} = (\Xi'\Xi)^{-1}\Xi'Y. \tag{6.3}$$

Now let $\hat{b}_{SE} = (\hat{b}_{SE}(-(M-1)/2), \ldots, \hat{b}_{SE}(0), \ldots, \hat{b}_{SE}((M-1)/2)'$ be the estimate of b defined by (2.8). Letting W be the $M \times M$ unitary matrix defined in (4.1), D_{XY} be the $M \times M$ diagonal matrix with r, rth element $\hat{f}^{XY}(2\pi r/M)$, and D_{XX} be the $M \times M$ diagonal matrix with r, rth element $\hat{f}^{XX}(2\pi r/M)$, it is easily checked that $\hat{b}_{SE}(r-s)$ is the r, sth element of the matrix $WD_{XX}^{-1}D_{XY}W^* = (WD_{XX}W^*)^{-1} WD_{XY}W^*$, and hence $\hat{b}_{SE}$ is the $(((M-1)/2)+1)$st column of this matrix.

Now let $\tilde{\Psi}$ be the $T \times M$ matrix with $Y(j-k+((M-1)/2)+1)$ in the j, kth position, where $Y(s) = Y(s+T)$ for $s < 1$, $Y(s) = Y(s-T)$ for $s > T$ and let $\tilde{\Xi}$ be the matrix gotten from Ξ by replacing $X(s)$ by $X(s+T)$ whenever $s < 1$ and by $X(s-T)$ whenever $s > T$. Then, approximately $\hat{b}_{LS}$ is the $(((M-1)/2)+1)$st column of the matrix

$$(\tilde{\Xi}'\tilde{\Xi})^{-1}\tilde{\Xi}'\tilde{\Psi}. \tag{6.4}$$

The r, sth element of $\tilde{\Xi}'\tilde{\Psi}$ is an estimate, say $\tilde{R}^{XY}(r-s)$ of $R^{XY}(r-s)$. (The entries of $\tilde{\Xi}'\tilde{\Psi}$ depend only on $r-s$.) Again approximately

$$\tilde{\Xi}'\tilde{\Psi} \approx W\tilde{D}_{XY}W^* \tag{6.5}$$

where $\tilde{D}_{XY}$ is the diagonal matrix with $\tilde{f}^{XY}(2\pi r/M)$ in the r, rth position, where $\tilde{f}^{XY}(\omega)$ is defined as

$$\tilde{f}^{XY}(\omega) = \frac{1}{2\pi} \sum_{\tau=-(M-1)}^{(M-1)} \tilde{R}^{XY}(\tau)\, e^{-i\omega\tau}. \tag{6.6}$$

This follows by observing that the r, sth element of the right hand side of (6.5) is given by $\tilde{R}^{XY}(\tau) + \tilde{R}^{XY}(\tau + M) + \tilde{R}^{XY}(\tau - M)$, defining $\tilde{R}^{XY}(\tau) = 0$ for $|\tau| \geqslant M$. Thus if $\tilde{R}^{XY}(\tau)$ is small for τ large, the matrices in (6.5) agree for "most" of their elements. Similarly $(\tilde{\Xi}'\tilde{\Xi})^{-1} \approx W\tilde{D}_{XX}^{-1}W^*$, where $\tilde{D}_{XX}$ is defined analogously to $\tilde{D}_{XY}$. The terms $\tilde{f}^{XY}(\omega)$ and $\tilde{f}^{XX}(\omega)$ are estimates of $f^{XY}(\omega)(f^{XY}(\omega) = f_{11}^{XY}(\omega))$, and $f^{XX}(\omega)$. Hence, to the extent that the end terms above are negligible and $\tilde{f}^{XY}(2\pi r/M) \approx \hat{f}^{XY}(2\pi r/M)$, $\tilde{f}^{XX}(2\pi r/M) \approx \hat{f}^{XX}(2\pi r/M)$, we have $\hat{b}_{LS} \approx \hat{b}_{SE}$.

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APPENDIX

The following theorem is proved in Wahba [**8**].

THEOREM: *Let $X(t)$ and $Y(t)$ be jointly stationary zero mean Gaussian time series possessing a spectral density matrix $F(\omega)$ strictly positive definite, all ω, with $\sum_{\tau=-\infty}^{\infty} |\tau|\,|R(\tau)| \leqslant \theta < \infty$ for each $R(\tau)$, $R(\tau)$ being the ξ, η, n*th *entry of $R^{XX}(\tau)$ or $R^{XY}(\tau)$ or $R^{YY}(\tau)$. Let the spectral density matrices $\hat{F}(2\pi v/M)$, $v = 1, 2, \ldots, M$, based on a record of length T, be estimated by non-overlapping averages of n neighboring periodograms, as given in* (3.1), *with $n \sim T/M$. Define, for each T, M,*

$$\bar{F}\left(\frac{2\pi v}{M}\right) = \frac{1}{n}\sum_{j=-(n-1)/2}^{(n-1)/2} F\left(\frac{2\pi v}{M} + 2\pi\frac{j}{T}\right) \qquad (v = 1, 2, \ldots, M).$$

Then, for each $T, M, n \approx T/2M$, a family $\tilde{F}(2\pi v/M)$, $v = 1, 2, \ldots M$, of complex Wishart matrices,

$\tilde{F}(2\pi v/M) \sim W_c(\bar{F}(2\pi v/M), P + Q, n)$ independent except that $\tilde{F}(2\pi v(M - v)/M) \equiv \tilde{F}^*(2\pi v/M)$, can be constructed on the sample space of $(X(t), Y(t), t = 1, 2, \ldots T)$ such that

$$\text{(A.1)} \qquad E \sum_{v=1}^{M} \textit{Trace} \left(\hat{F}\left(\frac{2\pi v}{M}\right) - \tilde{F}\left(\frac{2\pi v}{M}\right)\right)\left(\hat{F}\left(\frac{2\pi v}{M}\right) - \tilde{F}\left(\frac{2\pi v}{M}\right)\right)^{*\prime} \leqslant c_1 \frac{nM}{T^2} + c_2 \frac{\log M}{n^2},$$

where c_1 and c_2 are constants depending on θ and the largest and smallest eigenvalues of $F(\omega)$, but not n, M or T.

We remark that $\bar{F}(2\pi v/M)$ is an averaged version of $F(\omega)$ of (2.2) where the averaging takes place over the same frequencies for which the periodogram is averaged. Note that, since our conditions on $R(\tau)$ imply that the entries of $F(\omega)$ have bounded derivatives, $\bar{F}(2\pi v/M) \to F(2\pi v/M)$ as $n/T \approx 1/2M \to 0$.

Since the variance of the entries of $\tilde{F}(2\pi v/M)$ are proportional to $1/n$, letting $n = T/M$, we have, as $1/M \to 0$, $M \log M/T \to 0$, that the M random matrices of (3.2) converge jointly in mean square to independent complex Wishart matrices. We remark that the nature of the estimates, (3.1), i.e., averages of appropriately spared periodograms, enters into the proof of the above theorem in the following way: each entry in $\hat{F}(2\pi v/M)$ is a quadratic form in the observations, and, except for a constant multiple, the quadratic forms for distinct v form a family of orthogonal projections. It appears that this fact is required to obtain results this sharp. Heuristically speaking this corresponds to windowed estimates with non-overlapping "square" windows.

For ease of notation, and without loss of generality, let $P = Q = 1$; hence $\hat{b}(s)$, and $\tilde{b}(s)$, and $\hat{F}^{XX}(2\pi v/M)$, $\tilde{F}^{XX}(2\pi v/M)$, etc., are one dimensional, where $\hat{b}(s)$, $\tilde{b}(s)$, are defined as functions of $\hat{F}(\omega)$ and $\tilde{F}(\omega)$ by (2.8) and (3.2).

Now

$$\hat{b}(s) - \tilde{b}(s) = \frac{1}{M} \sum_{v=1}^{M} \left(\hat{B}\left(\frac{2\pi v}{M}\right) - \tilde{B}\left(\frac{2\pi v}{M}\right)\right) e^{-2\pi i v s/T},$$

and it is straight forward to show that

$$\text{(A.2)} \qquad |\hat{b}(s) - \tilde{b}(s)| \leqslant \frac{1}{M} \sum_{v=1}^{M} g_v \sum_{A,B=X,Y} \left| \hat{F}^{AB}\left(\frac{2\pi v}{M}\right) - \tilde{F}^{AB}\left(\frac{2\pi v}{M}\right)\right|$$

where g_v is a function of the entries of $\hat{F}(2\pi v/M)$ and $\tilde{F}(2\pi v/M)$ such that $Eg_v^2 \leqslant$ constant. Since the variance $\sigma^2(\tilde{b}(s))$ of $\tilde{b}(s) \approx$ const.$/nM$ by (3.4), we have

$$\frac{E|\hat{b}(s) - \tilde{b}(s)|}{\sigma(\tilde{b}(s))} \leqslant \text{const.}\sqrt{\frac{n}{M}} \sum_{v=1}^{M} [Eg_v^2]^{\frac{1}{2}} \sum_{A,B=X,Y} \left\{E\left|\hat{F}^{AB}\left(\frac{2\pi v}{M}\right) - \tilde{F}^{AB}\left(\frac{2\pi v}{M}\right)\right|^2\right\}^{\frac{1}{2}}$$

$$\leqslant \text{const.}\ \sqrt{n}\left[\sum_{v=1}^{M} \sum_{A,B=X,Y} E\left|\hat{F}^{AB}\left(\frac{2\pi v}{M}\right) - \tilde{F}^{AB}\left(\frac{2\pi v}{M}\right)\right|^2\right]^{\frac{1}{2}}$$

which, by (A.1)

$$\leqslant \text{const.}\ \left(\frac{M \log M}{T} + \frac{Mn^2}{T^2}\right)^{\frac{1}{2}}$$

$$\leqslant \text{const.}\ \left(\frac{M \log M}{T} + \frac{1}{M}\right)^{\frac{1}{2}}.$$

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