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REGULARIZATION AND APPROXIMATION OF LINEAR OPERATOR EQUATIONS IN REPRODUCING KERNEL SPACES[1]

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1. **Introduction**. Let X and Y be real Hilbert spaces and let A be a linear operator with domain $\mathcal{D}(A) \subset X$ and range in Y. An element $u \in \mathcal{D}(A)$ is said to be a *least-squares* solution of the equation

$$(1) \qquad Ax = y$$

for a given $y \in Y$ if $\inf\{\|Ax - y\| : x \in \mathcal{D}(A)\} = \|Au - y\|$. A *pseudo-solution* of (1) for a given $y \in Y$ is a least-squares solution of minimal norm. Equation (1) *is well-posed* relative to the spaces X, Y if for each $y \in Y$, (1) has a unique pseudosolution which depends continuously on y; otherwise the equation is said to be *ill-posed.*

One objective of this research is to show, when X and Y are L_2-spaces of square-integrable functions, that the topology of reproducing kernel Hilbert spaces (RKHS) is an appropriate topology for the regularization of ill-posed linear operator equations, and to initiate a study of generalized inverses of linear operators acting between two RKHS. A second objective is to provide an approach to optimal approximations of linear operator equations in the context of RKHS, and to demonstrate the relation between the regularization operator of the equation $Af = g$ and the generalized inverse of A in an appropriate RKHS. (For some background on regularization methods see [3], [5], [9]; for generalized inverses see, for example, [4].)

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2. **Generalized inverses in RKHS.** Let H_Q denote the RKHS of real-valued functions on the bounded interval S with reproducing kernel $Q(s, s')$. Denote the inner product and norm in H_Q by $\langle \cdot , \cdot \rangle_Q$ and $\| \cdot \|_Q$ respectively. Then for $f \in H_Q, f(s) = \langle Q_s, f \rangle_Q$ where $Q_s(s') := Q(s, s') = \langle Q_s, Q_{s'} \rangle$ for $s, s' \in S$. (See [1] for properties of RKHS.) The kernel $Q(s, s')$ induces a selfadjoint Hilbert-Schmidt operator on $L_2[S]$, the space of square-integrable real-valued functions on S, by

$$(Qf)(s) = \int_S Q(s, s')f(s')ds'.$$

Furthermore, we have $H_Q = Q^{1/2}(L_2[S])$ and

$$\|f\|_Q = \inf\{ \|p\|_{L_2[S]} : p \in L_2[S], Q^{1/2}p = f \}$$

(cf. [8], [10]). For $f \in H_Q$, let $Q^{-1/2}f$ denote the element p of minimal $L_2[S]$-norm that satisfies $Q^{1/2}p = f$. We then have

$$\langle f_1, f_2 \rangle_Q = \langle Q^{-1/2}f_1, Q^{-1/2}f_2 \rangle_{L_2[S]}.$$

THEOREM 1. *Let A be a linear operator from $X = L_2[S]$ into $Y = L_2[T]$, where S, T are closed bounded intervals. Assume that A has the following properties*:

(i) $H_Q \subset \mathcal{D}(A) \subset X$ *(throughout "$\subset$" denotes point-set inclusion only), where H_Q is an RKHS with continuous kernel on $S \times S$*;

(ii) $A[H_Q] \subset H_R \subset H_{\widetilde{R}} \subset Y$, *where H_R and $H_{\widetilde{R}}$ are RKHS with continuous kernels on $T \times T$; and*

(iii) *the null space of A in H_Q is closed in H_Q.*

Let $A^\dagger_{X,Y}$ (resp. $A^\dagger_{Q,\widetilde{R}}$) denote the generalized inverse of A when A is considered as a map from X into Y (resp. from H_Q into $H_{\widetilde{R}}$). Let $y \in \mathcal{D}(A^\dagger_{Q,\widetilde{R}})$. Then $y \in \mathcal{D}((Q^{1/2}AQ^{1/2})^\dagger_{(X,Y)}\widetilde{R}^{-1/2})$ and

$$A^\dagger_{(Q,\widetilde{R})} y = Q^{1/2}(\widetilde{R}^{-1/2}AQ^{1/2})^\dagger_{X,Y}\widetilde{R}^{-1/2}y. \tag{2}$$

(The operators Q and $\widetilde{R}$ are those induced by the RKHS H_Q and $H_{\widetilde{R}}$ respectively.)

It should be noted that an operator A may map H_Q onto another RKHS, while failing to have a closed range in $L_2[T]$. This is, for example, the case if A is a Hilbert-Schmidt integral operator (with nondegenerate kernel) on $L_2[S]$.

3. **Explicit representations of minimal-norm solutions of linear operator equations in RKHS.** We assume that H_Q is chosen so that

(3) the linear functionals E_t defined by $E_t f = (Af)(t)$ are continuous in H_Q.

Then there exists $\eta_t \in H_Q$ for $t \in T$ such that $(Af)(t) = \langle \eta_t, f\rangle_Q$, where $\eta_t(s) = \langle \eta_t, Q_s\rangle_Q = AQ_s(t)$. Let H_R be the RKHS with kernel $R(t, t') := \langle \eta_t, \eta_{t'}\rangle_Q$, $t, t' \in T$. Let V be the closure of the span of $\{\eta_t : t \in T\}$ in H_Q. It follows easily that the null space of A *in* H_Q is $V^\perp$ ($\perp$ in H_Q). Since $\langle \eta_t, \eta_{t'}\rangle_Q = \langle R_t, R_{t'}\rangle_R$, where $R_t(t') := R(t, t')$, there is an *isometric isomorphism* between the subspace V and H_R generated by the correspondence $\eta_t \in V \sim R_t \in H_R$. Under this isomorphism, $f \sim g \Longleftrightarrow \langle \eta_t, f\rangle_Q = \langle R_t, g\rangle$, i.e., $g(t) = (Af)(t)$; $P_V Q_s \sim \eta_s^* := AQ_s$, where P_V is the orthogonal projector from H_Q onto V.

For $g \in H_R$ let $\hat{f}$ be the element in H_Q of minimal H_Q-norm which satisfies the equation $Af = g$. Then $\hat{f} \in V$ and $g \sim \hat{f}$. We have the following representations for $\hat{f}$.

THEOREM 2. *If* (3) *holds and* $g \in H_R$, *then* $\hat{f}(s) = \langle Q_s, \hat{f}\rangle_Q = \langle \eta_s^*, g\rangle_R$. *Furthermore, if* $\mathcal{D}(A^*)$ *is dense in* Y, *where* A^* *is the adjoint of* A *considered as an operator from* X *into* Y, *and if* H_Q, $H_R = A(H_Q)$ *possess continuous kernels, then* $A^\dagger_{(Q,R)} g = QA^*(AQA^*)^\dagger_{Y,Y}\, g$.

4. **Regularization in RKHS.** Let H_Q and H_P be RKHS with norms $\|\cdot\|_Q$ and $\|\cdot\|_P$ respectively. By a *regularized pseudosolution* (in RKHS) of the equation $Af = g$, we mean a solution to the variational problem: Find $f_\lambda \in H_Q$ to minimize

$$\phi_g(f) = \|Af - g\|_P^2 + \lambda \|f\|_Q^2, \qquad \lambda > 0 \tag{4}$$

($\phi_g(f)$ will be assigned $+\infty$ if $Af - g \notin H_P$). In this section, A is a linear operator densely defined on $L_2[S]$ into $L_2[T]$, and H_Q must be chosen so that $A[H_Q] = H_R$, where H_R is some RKHS contained (as a set) in $L_2[T]$, and H_P is a subset of $L_2[T]$. Assume $g = g_0 + \xi$ for some $g_0 \in A[H_Q]$ and some $\xi \in H_P$. For $\lambda > 0$, let $H_{\lambda P}$ be the RKHS with kernel $\lambda P(t, t')$, where $P(t, t')$ is the (continuous) kernel associated with H_P. We have $H_P = H_{\lambda P}$ and $\|\cdot\|_P^2 = \lambda \|\cdot\|_{\lambda P}^2$. Let $R(\lambda) = R + \lambda P$, and let $H_{R(\lambda)}$ be the RKHS with kernel $R(\lambda; t, t')$. Then (see Aronszajn [**1**, p. 352]) $H_{R(\lambda)}$ is the Hilbert space of functions of the form $g = g_0 + \xi$, where $g_0 \in H_R$ and $\xi \in H_P$. Following Aronszajn [1], we note that this decomposition is not unique unless $H_P \cap H_R = \{0\}$. The norm on $H_{R(\lambda)}$ is given by

$$\|g\|^2_{R(\lambda)} = \min \{\|g_0\|^2_R + \|\xi\|^2_{\lambda P} : g_0 \in H_R, \xi \in H_P, g_0 + \xi = g\}.$$

THEOREM 3. *Suppose* $\mathcal{D}(A^*)$ *is dense in* Y, $H_Q \subset \mathcal{D}(A)$ *and* A *and* H_Q *satisfy* (3). *Suppose* H_Q, H_R *and* $H_P \subset Y$ *all have continuous kernels. Then for* $g \in H_{R(\lambda)}$, *the unique minimizing element* $f_\lambda \in H_Q$ *of the functional* $\phi_g(f)$ *is given by*

$$f_\lambda(s) = \langle AQ_s, g\rangle_{R(\lambda)} = (QA^*(AQA^* + \lambda P)^\dagger_{Y,Y}\, g)(s).$$

We call the (linear) mapping which assigns to each $g \in H_{R(\lambda)}$ the unique minimizing element f_λ, the *regularization operator* of the equation $Af = g$.

THEOREM 4. *If* $H_P \cap H_R = \{0\}$, *then the minimizing element* f_λ *of* (4) *is the solution to the problem: Find* $f \in \Omega$ *to minimize* $\|f\|_Q$, *where*

$$\Omega = \left\{ f \in H_Q : \|Af - g\|_{R(\lambda)} = \inf_{h \in H_Q} \|Ah - g\|_{R(\lambda)} \right\}.$$

In the setting of this section we have $A[H_Q] = H_R \subset H_{R(\lambda)} \subset Y$. Replacing $H_{\tilde{R}}$ by $H_{R(\lambda)}$ in (2), we obtain

$$A^\dagger_{(Q,R(\lambda))}y = Q^{1/2}[(R + \lambda P)^{-1/2}AQ^{1/2}]^\dagger_{(X,Y)}(R + \lambda P)^{-1/2}y$$

for $y \in \mathcal{D}(A^\dagger_{(Q,R(\lambda))})$. It is helpful to remember that the topology on H_R is not, in general, the restriction of the topology of $H_{R+\lambda P}$, with the notable exception of the case $H_R \cap H_P = \{0\}$. The authors provide elsewhere concrete examples arising in approximation of boundary-value problems where H_R is not a closed subspace of $H_{R+\lambda P}$.

We emphasize that if $H_R \cap H_P = \{0\}$, then H_R is a closed subspace of $H_{R+\lambda P}$; in this case the regularization operator is a generalized inverse in an appropriate RKHS (Theorem 4).

5. **Convergence rates of approximate regularized solutions to linear operator equations.** The regularization method of §4 requires in practice some approximate procedure for solving (4) numerically. The principal result of [7] is the establishment of uniform *pointwise* convergence of approximate

solutions obtained by moment discretization of (4). Let $T_n = \{t_1, t_2, \cdots, t_n\}$, where $t_i \in T$, $t_1 < t_2 < \cdots < t_n$. For a generic function h on T, let $h_n = (h(t_1), \cdots, h(t_n))$. Let P_n denote the $n \times n$ matrix whose ijth element is $P(t_i, t_j)$, and define $\|h_n\|_{P_n} = \min\{\|e\| : e \in R^n, P_n^{1/2}e = h_n\}$, if $h_n \in R(P_n)$; ∞ otherwise.

THEOREM 5. *Let the operator A be as in* §4. *Let $f_{\lambda,n}$ be the minimizing element in $\mathcal{H}_Q$ of the functional $J_n = \|(Af)_n - g_n\|_{P_n}^2 + \lambda\|f\|_Q^2$ for $\lambda > 0$. Let $P_{T_n}(\lambda)$ be the orthogonal projector of $\mathcal{H}_{R(\lambda)}$ onto the subspace spanned by $\{R_t(\lambda) : t \in T_n\}$. Then*

$$|f_\lambda(s) - f_{\lambda,n}(s)| = |\langle P_{T_n}(\lambda)\eta_s^*, P_{T_n}(\lambda)g\rangle_{R(\lambda)}|$$

$$\leqslant \|\eta_s^* - P_{T_n}(\lambda)\eta_s^*\|_{R(\lambda)}\|g - P_{T_n}(\lambda)g\|_{R(\lambda)}.$$

Furthermore, let $\Delta = \max|t_{i+1} - t_i|$, $|f_\lambda(s) - f_{\lambda,n}(s)| = O(\Delta^m)$ or $O(\Delta^{2m})$, depending on smoothness properties of the kernel $R(\lambda, t, t')$ and the functions g and η_s^*. In the particular case when $\mathcal{H}_P \cap \mathcal{H}_R = \{0\}$, $\{f_{\lambda,n}\}$ converges to $A^\dagger g$. (An explicit formula for $f_{\lambda,n}$ and further properties of f_λ are given in [7].)

The results of this paper provide an approach to simultaneous regularization and approximation of (ill-posed) linear operator equations which applies to a large class of operator equations that include boundary-value problems, Fredholm integral equations of the first kind, and integrodifferential equations; see also [2], [10]. The Sobolev spaces W_2^m, $m \geqslant 0$ are included in the class of spaces considered.

The proofs of the preceding theorems as well as related results and examples will appear in [6] and [7].

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