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GENERALIZED INVERSES IN REPRODUCING KERNEL SPACES: AN APPROACH TO REGULARIZATION OF LINEAR OPERATOR EQUATIONS*

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Abstract. In this paper a study of generalized inverses of linear operators in reproducing kernel Hilbert spaces (RKHS) is initiated. Explicit expressions for generalized inverses and minimal-norm solutions of linear operator equations in RKHS are obtained in several forms. The relation between the regularization operator of the equation $Af = g$ and the generalized inverse of the operator A in RKHS is demonstrated. In particular, it is shown that they are the same if the range of the operator is closed in an appropriate RKHS. Finally, properties of the regularized pseudosolutions in this setting are studied.

It is shown that this approach provides a natural and effective setting for regularization problems when the operator maps one RKHS into another.

1. Introduction. Let X and Y be Hilbert spaces and let A be a linear operator on a domain $\mathscr{D}(A) \subset X$ into Y. The operator A is said to have a generalized inverse $A^\dagger$ on a domain $\mathscr{D}(A^\dagger) \subset Y$ if for each $y \in \mathscr{D}(A^\dagger)$, $\inf\{\|Ax - y\| : x \in X\} = \|AA^\dagger y - y\|$ and $\|A^\dagger y\|$ is smaller than the norm of any other element $u \in X$ at which the preceding infimum is attained. It is well known and can be easily shown that if A is a bounded operator, or if A is a densely defined closed operator, then $A^\dagger$ exists on $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, where $\mathscr{R}(A)$ is the range of A. The domain $\mathscr{D}(A^\dagger)$ in this case is a dense subset of Y and $A^\dagger$ is unbounded unless $\mathscr{R}(A)$ is closed in Y. A compact operator with infinite-dimensional range is a prototype of an operator for which $\mathscr{R}(A)$ is not closed.

To impart continuity to $A^\dagger$ when $\mathscr{R}(A)$ is not closed in Y, one might consider subsets X', Y' of X, Y, respectively, equipped with topologies which are not equivalent to those of X and Y, and such that the generalized inverse of A, when viewed as an operator from Y' to X', exists and is bounded. The topologies of X' and Y' are required to be induced by inner products, and must be amenable to the original setting of the operator equation $Ax = y$, so that questions of least squares solvability and related approximation schemes are still meaningful in a wide context.

One objective of this paper is to show, when X and Y are $\mathscr{L}_2$-spaces of square-integrable real-valued functions, that the topology of reproducing kernel spaces is an appropriate topology for the goal stated above, and thereby to initiate a systematic study of generalized inverses of linear operators acting between two reproducing kernel Hilbert spaces. This study has strong interface with the problem of regularization of (ill-posed or poorly-conditioned) linear operator equations. This brings us to another objective of this paper, which is to provide a new approach to regularization in the context of RKHS.

At present there are several approaches to the investigation and regularization of ill-posed problems. These are discussed briefly in our report [10], which forms an earlier draft of this paper and contains an extensive bibliography on these

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approaches. In this paper we present another approach to regularization based on the notion of least squares solution of minimal norm and on regularization operators in RKHS. Our approach coincides in philosophy with some of the known approaches cited in [5], [16], [10] (in the sense that we change the notion of the solution and consider the problem in new spaces), even though it differs sharply in technical details. We exploit (in an optimal way) the geometry of RKHS and obtain results which are the best possible in this context. The basic results of this paper are stated in Theorems 3.1, 4.1, 4.2, 5.1, 5.2 and 6.1. Applications of this approach to rates of convergence of approximate solutions will appear elsewhere [11], [12].

To our knowledge this is the first time that generalized inverses of linear operators and reproducing kernels are used *simultaneously* in the same context. It is befitting to mention here that the concepts of a generalized inverse (of a matrix) and RKHS both go back to the work of E. H. Moore [7].

2. Generalized inverses, reproducing kernel spaces, and pseudosolutions of linear operator equations. Let X and Y be two Hilbert spaces over the real scalars and let A be a linear operator on $\mathscr{D}(A) \subset X$ into Y. Let $\mathscr{R}(A)$, $\mathscr{N}(A)$ and A^* denote, respectively, the range, nullspace and adjoint of A. The orthogonal compliment of a subspace S is denoted by $S^{\perp}$; the closure of S is denoted by $\bar{S}$ and the orthogonal projector on a closed subspace $\mathscr{M}$ is denoted by $P_{\mathscr{M}}$.

We consider the linear operator equation

$$Ax = y. \tag{2.1}$$

DEFINITION 2.1. An element $u \in X$ is said to be a *least squares solution* of (2.1) if $\inf \{\|Ax - y\| : x \in X\} = \|Au - y\|$. If the set S_y of all least squares solutions of (2.1) for a given $y \in Y$ has an element v of minimal norm, then v is called a *pseudosolution* of (2.1).

DEFINITION 2.2. The operator equation (2.1) is said to be *well-posed* (relative to the spaces X and Y) if for each $y \in Y$, (2.1) has a unique pseudosolution which depends continuously on y; otherwise the equation is said to be *ill-posed.*

Obviously (2.1) has a least squares solution for a given $y \in Y$ if and only if there exists an element $w \in \mathscr{R}(A)$ which is closest to y. From this it follows immediately that (2.1) has a least squares solution if and only if $P_{\overline{\mathscr{R}(A)}}\, y \in \mathscr{R}(A)$, or equivalently $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^{\perp}$. For such y, it is easy to see that the set S_y has a unique element of minimal norm if and only if $P_{\overline{\mathscr{N}(A)}}\, u \in \mathscr{N}(A)$ for some $u \in S_y$ (in which case this is also true for each $x \in S_y$). Thus a pseudosolution of (2.1) exists if and only if

$$y \in A(\mathscr{D}(A) \cap \mathscr{N}(A)^{\perp}) \oplus \mathscr{R}(A)^{\perp}. \tag{2.2}$$

In what follows we shall primarily be interested in the cases when A is a *closed* linear operator on a dense domain $\mathscr{D}(A) \subset X$, *or* when A is a *bounded* linear operator on X. In either of these cases, since $\mathscr{N}(A)$ is closed, condition (2.2) reduces to the condition

$$y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^{\perp}. \tag{2.3}$$

The (linear) map which associated with each y satisfying (2.3) a unique pseudosolution defines the *generalized inverse* of A, which is denoted by $A^{\dagger}$.

For each $y \in \mathscr{D}(A^\dagger)$, we thus have $S_y = A^\dagger y \oplus \mathscr{N}(A)$. Note that in our setting, $A^\dagger$ is a densely defined operator.

We summarize in the following proposition equivalent properties of the generalized inverse (see Nashed [8]).

PROPOSITION 2.1. *Each of the following sets of conditions characterizes the generalized inverse $A^\dagger$ of a bounded or a densely defined closed operator*:

(a) $AA^\dagger A = A$ *on* $\mathscr{D}(A)$, $A^\dagger AA^\dagger = A^\dagger$ *on* $\mathscr{D}(A^\dagger)$, $AA^\dagger = P_{\overline{\mathscr{R}(A)}}|\mathscr{D}(A^\dagger)$ *and* $A^\dagger A = P_{\mathscr{N}(A)^\perp}|\mathscr{D}(A)$, *where the vertical bar denotes the restriction of the projector to the indicated domain.*

(b) $A^\dagger$ *is the unique linear extension of* $\{A|\mathscr{N}(A)^\perp\}^{-1}$ *to* $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$ *so that* $\mathscr{N}(A^\dagger) = \mathscr{R}(A)^\perp$.

(c) *For* $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, $A^\dagger y$ *is the unique solution of minimal norm of the "normal" equation* $A^*Ax = A^*y$, *provided* $\mathscr{R}(A) \subset \mathscr{D}(A^*)$.

(d) *For* $y \in \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, $A^\dagger y$ *is the unique solution of minimal norm of the "projectional" equation* $Ax = P_{\overline{\mathscr{R}(A)}}y$.

PROPOSITION 2.2 *The following statements are equivalent for A as above*:

(a) *The operator equation* (2.1) *is well-posed in* (X, Y).

(b) *A has a closed range in Y.*

(c) $A^\dagger$ *is a bounded operator on Y into X.*

Proof. (a) implies that $\mathscr{D}(A^\dagger) = Y$ and thus from (2.3), $\mathscr{R}(A) = \overline{\mathscr{R}(A)}$. Statement (c) follows from (b) using Proposition 2.1(b) and the closed graph theorem. That (c) implies (a) is obvious.

Convention 2.1. In this paper we encounter on several occasions a composition of two operators, say A and B, where B is unbounded and densely defined but AB is bounded. In all such cases we shall assume that AB has already been *extended* as usual (i.e., by continuity) to the *closure* of the domain of B. An example is the composition $AA^\dagger$ when $\mathscr{R}(A)$ is a nonclosed subspace. Then $\mathscr{D}(A^\dagger)$ is dense, but $AA^\dagger$ is bounded and can be extended to $\overline{\mathscr{R}(A)} \oplus \mathscr{R}(A)^\perp$, even though $A^\dagger$ cannot (see also part (a) of Proposition 2.1).

When $\mathscr{R}(A)$ is not closed, the problem of finding least squares solutions of (2.1) is ill-posed relative to the spaces X, Y. An ill-posed problem relative to (X, Y) may be recast in some cases as a well-posed problem relative to new spaces $X' \subset X$ and $Y' \subset Y$, with topologies on X' and Y' which are different respectively from the topologies on X and Y. From the point of regularization, the topologies on X' and Y' should not be too restrictive and must lend themselves to requirements which are satisfied by a wide class of admissible solutions of pseudosolutions. This is precisely the point which we exploit in connection with the topologies on reproducing kernel Hilbert spaces.

A Hilbert space $\mathscr{H}$ of real-valued functions defined on a set S is said to be a *reproducing kernel Hilbert space* (RKHS) if all the evaluation functionals $f \to f(s)$ for $f \in \mathscr{H}$ and $s \in S$ are continuous. In this case there exists, by the Riesz representation theorem, a unique element in $\mathscr{H}$ (call it Q_s) such that

$$(2.4) \qquad \langle f, Q_s\rangle = f(s), \qquad f \in \mathscr{H}.$$

The *reproducing kernel* (RK) is defined by

$$(2.5) \qquad Q(s, s') := \langle Q_s, Q_s'\rangle, \qquad s, s' \in S.$$

Let $\mathscr{H}_Q$ denote the RKHS with reproducing kernel Q, and denote the inner product and norm in $\mathscr{H}_Q$ by $\langle \cdot, \cdot \rangle_Q$ and $\| \cdot \|_Q$, respectively. Note that $Q(s, s')$ ($\equiv Q_s(s')$) is a nonnegative definite symmetric kernel on $S \times S$, and that $\{Q_s, s \in S\}$ spans $\mathscr{H}_Q$ since $\langle Q_s, f \rangle_Q = 0$, $s \in S$, implies $f(s) = 0$. For properties of reproducing kernel spaces, see Aronszajn [1], Shapiro [15, Chap. 6] and Parzen [13].

If S is a bounded interval (or if S is an unbounded interval but $\iint Q^2(s, s')\, ds\, ds' < \infty$), and $Q(s, s')$ is continuous on $S \times S$ (the only case we shall consider here), then it is easy to show that $\mathscr{H}_Q$ is a space of continuous functions. Note also that $\mathscr{L}_2[S]$ is *not* an RKHS since the evaluation functionals are not continuous.

An RKHS $\mathscr{H}_Q$ with RK Q determines a self-adjoint Hilbert–Schmidt operator (also denoted by Q) on $\mathscr{L}_2[S]$ to $\mathscr{L}_2[S]$ by

$$(Qf)(s) = \int_S Q(s, s') f(s')\, ds', \qquad f \in \mathscr{L}_2[S]. \tag{2.6}$$

Since $Q(s, s')$ is assumed to be continuous, then by the theorems of Mercer, Hilbert and Schmidt [14, pp. 242–246], the operator Q has an $\mathscr{L}_2[S]$-complete orthonormal system of eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ with $\lambda_i \geqq 0$ and $\sum_{i=1}^{\infty} \lambda_i < \infty$ (thus Q is a *trace-class* operator; see [2, Chap. XI.9] or [3, Chap. 2]); also $Q(s, s')$ has the uniformly convergent Fourier expansions

$$Q(s, s') = \sum_{i=1}^{\infty} \lambda_i\, \phi_i(s) \phi_i(s')$$

and

$$Qf = \sum_{i=1}^{\infty} \lambda_i (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i, \tag{2.7}$$

where $(\cdot, \cdot)_{\mathscr{L}_2[S]}$ is the inner product in $\mathscr{L}_2[S]$.

It is well known (see, for example, [17]) that

$$\mathscr{H}_Q = \{f : f \in \mathscr{L}_2[S],\ \sum_{i=1}^{\infty} \lambda_i^{-1} (f, \phi_i)^2_{\mathscr{L}_2[S]} < \infty\},$$

where the notational convention $0/0 = 0$ is being adopted, and

$$\langle f_1, f_2 \rangle_Q = \sum_{i=1}^{\infty} \lambda_i^{-1} (f_1, \phi_i)_{\mathscr{L}_2[S]} (f_2, \phi_i)_{\mathscr{L}_2[S]}.$$

The operator Q has a well-defined symmetric square root $Q^{1/2}$ which is a Hilbert–Schmidt operator ([14, pp. 242–246] or [3, Chap. 2]):

$$Q^{1/2} f = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i. \tag{2.8}$$

Thus, since $\mathscr{N}(Q) = \mathscr{N}(Q^{1/2})$,

$$\mathscr{H}_Q = Q^{1/2}(\mathscr{L}_2[S]) = Q^{1/2}(\mathscr{L}_2[S] \ominus \mathscr{N}(Q)).$$

$(Q^{1/2})^\dagger$ has the representation

$$(2.9) \qquad (Q^{1/2})^\dagger f = \sum_{i=1}^{\infty} (\sqrt{\lambda_i})^\dagger (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i$$

on $\mathscr{H}_Q \oplus \mathscr{H}_Q^\perp$ ($\perp$ in $\mathscr{L}_2[S]$), where, for θ a real number, $\theta^\dagger = \theta^{-1}$, $\theta \neq 0$; $\theta^\dagger = 0$, $\theta = 0$. Similarly $Q^\dagger$ has the representation

$$(2.10) \qquad Q^\dagger f = \sum_{i=1}^{\infty} \lambda_i{}^\dagger (f, \phi_i)_{\mathscr{L}_2[S]} \phi_i$$

on its domain.

For any operator Q on $\mathscr{L}_2[S]$ induced by an RK $Q(s, s')$, as in (2.6) we shall adopt the notational conventions

$$(2.11) \qquad Q^{-1/2} := (Q^{1/2})^\dagger \quad \text{and} \quad Q^{-1} := Q^\dagger.$$

We have the relations

$$\|f\|_Q = \inf\{\|p\|_{\mathscr{L}_2[S]}, p \in \mathscr{L}_2[S], f = Q^{1/2}p\}, \qquad f \in \mathscr{H}_Q,$$

$$\langle f_1, f_2\rangle_Q = (Q^{-1/2}f_1, Q^{-1/2}f_2)_{\mathscr{L}_2[S]}, \qquad f_1, f_2 \in \mathscr{H}_Q,$$

and, if $f_1 \in \mathscr{H}_Q$ and $f_2 \in \mathscr{H}_Q$ with $f_2 = Q\rho$ for some $\rho \in \mathscr{L}_2[S]$, then

$$(2.12) \qquad \langle f_1, f_2\rangle_Q = (f_1, \rho)_{\mathscr{L}_2[S]}.$$

3. Relationship between generalized inverses in RKHS and $\mathscr{L}_2$-spaces. We are now ready to explore properties of the generalized inverse of a linear operator between two RK spaces. In the remainder of this paper we let $X = \mathscr{L}_2[S]$ and $Y = \mathscr{L}_2[T]$ denote the Hilbert spaces of square-integrable real-valued functions on the closed, bounded intervals S and T, respectively. Let A be a linear operator from X into Y. Let $\subset$ denote point set inclusion only, and suppose that A has the following properties:

$$(3.1) \qquad \mathscr{H}_Q \subset \mathscr{D}(A) \subset X,$$

where $\mathscr{H}_Q$ is an RKHS with continuous RK on $S \times S$;

$$(3.2) \qquad A(\mathscr{H}_Q) = \mathscr{H}_{\tilde{R}} \subset \mathscr{H}_R \subset Y,$$

where $\mathscr{H}_{\tilde{R}}$ and $\mathscr{H}_R$ are RKHS with continuous RK's on $T \times T$; and

$$(3.3) \qquad \mathscr{N}(A) \text{ in } \mathscr{H}_Q \text{ is closed in } \mathscr{H}_Q.$$

We emphasize in particular that the space $\mathscr{H}_{\tilde{R}}$ is not necessarily closed in the topology of $\mathscr{H}_R$.

Let $A^\dagger_{(X,Y)}$ denote the generalized inverse of A, when A is considered as a map from X into Y, and let $A^\dagger_{(Q,R)}$ denote the generalized inverse of A when A is considered as a map from $\mathscr{H}_Q$ into $\mathscr{H}_R$. Now the topologies in (X, Y) are not the same as the topologies in $(\mathscr{H}_Q, \mathscr{H}_R)$. Thus the generalized inverses $A^\dagger_{(X,Y)}$ and $A^\dagger_{(Q,R)}$ have distinct continuity properties in general. We shall now develop the relation between $A^\dagger_{(Q,R)}$ and certain (X, Y) and (Y, Y) generalized inverses. In the sequel, the operators $R: Y \to Y$ and $R^{1/2}: Y \to Y$ are defined from the RK of $\mathscr{H}_R$ analogous to Q and $Q^{1/2}$; see (2.7) and (2.8). We continue the notational convention of (2.11), that is, $R^{-1} = R^\dagger = R^\dagger_{(Y,Y)}$ and $R^{-1/2} = (R^{1/2})^\dagger_{(Y,Y)}$.

THEOREM 3.1. *Under assumptions* (3.1)–(3.3), *let* $y \in \mathscr{D}(A^\dagger_{(Q,R)})$, *i.e.,* $y \in \mathscr{H}_{\tilde{R}} \oplus \mathscr{H}^\perp_{\tilde{R}}$ ($\perp$ *in* $\mathscr{H}_R$). *Then*

$$y \in \mathscr{D}(Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}) \tag{3.4}$$

and

$$A^\dagger_{(Q,R)}y = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}y. \tag{3.5}$$

Proof. The (maximal) domain of $A^\dagger_{(Q,R)}$ is $\mathscr{H}_{\tilde{R}} \oplus \mathscr{H}^\perp_{\tilde{R}}$ ($\perp$ in $\mathscr{H}_R$). Denote the operator $Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}$ by L. We first show that $\mathscr{D}(A^\dagger_{(Q,R)}) \subset \mathscr{D}(L)$. Let $\tilde{A} = R^{-1/2}AQ^{1/2}$. The operator $\tilde{A}$ is defined over all of X since $\mathscr{D}(Q^{1/2}) = X$, $Q^{1/2}(X) = \mathscr{H}_Q$, $\mathscr{D}(A) \supset \mathscr{H}_Q$, $A(\mathscr{H}_Q) = \mathscr{H}_{\tilde{R}} \subset \mathscr{H}_R$ and $\mathscr{H}_R \subset \mathscr{D}(R^{-1/2})$. Also $\mathscr{R}(\tilde{A}) = \tilde{A}(X) = R^{-1/2}(\mathscr{H}_{\tilde{R}}) \subset R^{-1/2}(\mathscr{H}_R) \subset Y$. Thus $\mathscr{D}(\tilde{A}^\dagger_{(X,Y)}) = \mathscr{R}(\tilde{A}) \oplus \mathscr{R}(\tilde{A})^\perp$ ($\perp$ in Y), and

$$\mathscr{D}(\tilde{A}^\dagger_{(X,Y)}) = R^{-1/2}(\mathscr{H}_{\tilde{R}}) \oplus (R^{-1/2}(\mathscr{H}_{\tilde{R}}))^\perp \quad (\perp \text{ in } Y).$$

We now show

$$y \in \mathscr{H}_{\tilde{R}} \quad \text{implies} \quad y \in \mathscr{D}(L), \tag{3.6}$$

$$y \in \mathscr{H}^\perp_{\tilde{R}} \quad (\perp \text{ in } \mathscr{H}_R) \quad \text{implies} \quad y \in \mathscr{D}(L). \tag{3.7}$$

To prove (3.6), let $y \in \mathscr{H}_{\tilde{R}}$. Then $R^{-1/2}y \in \mathscr{D}(\tilde{A}^\dagger_{(X,Y)})$, so $y \in \mathscr{D}(\tilde{A}^\dagger_{(X,Y)}R^{-1/2})$, which implies $y \in \mathscr{D}(L)$ since $\mathscr{R}(\tilde{A}^\dagger_{(X,Y)})$ is contained in X, the domain of $Q^{1/2}$. To prove (3.7), let $y \in \mathscr{H}^\perp_{\tilde{R}}$ ($\perp$ in $\mathscr{H}_R$). This means that $y \in \mathscr{D}(R^{-1/2})$ and $\langle y, g\rangle_R = 0$ for all $g \in \mathscr{H}_{\tilde{R}}$. But for each $g \in \mathscr{H}_{\tilde{R}}$ there exists a unique $\psi \in Y \ominus \mathscr{N}(R^{1/2})$ such that $g = \tilde{R}^{1/2}\psi$. Thus $(R^{-1/2}y, R^{-1/2}\tilde{R}^{1/2}\psi)_Y = 0$ for all $\psi \in Y \ominus \mathscr{N}(\tilde{R}^{1/2})$. Thus $R^{-1/2}y$ is orthogonal to $R^{-1/2}(\mathscr{H}_{\tilde{R}})$ in Y, so that $R^{-1/2}y \in (R^{-1/2}(\mathscr{H}_{\tilde{R}}))^\perp$, $\perp$ in Y, $y \in \mathscr{D}(\tilde{A}^\dagger_{(X,Y)}R^{-1/2})$ and hence in $\mathscr{D}(Q^{1/2}\tilde{A}^\dagger_{(X,Y)}R^{-1/2})$.

Now we prove (3.5). For $y \in \mathscr{D}(A^\dagger_{(Q,R)})$, let $z = A^\dagger_{(Q,R)}y$. Then z is the unique element of minimal $\mathscr{H}_Q$-norm in the set

$$\mathscr{S} = \{u: \|Au - y\|_R = \inf_{x \in \mathscr{H}_R} \|Ax - y\|_R\}. \tag{3.8}$$

Let $x = Q^{1/2}p$ for $p \in X$, and let $\tilde{y} = R^{-1/2}y$. Let

$$W = \{w: \|\tilde{A}w - \tilde{y}\|_Y = \inf_{p \in X} \|\tilde{A}p - \tilde{y}\|_Y\}.$$

Then also

$$\begin{aligned} W &= \{w: \|R^{-1/2}AQ^{1/2}w - R^{-1/2}y\|_Y = \inf_{p \in X} \|R^{-1/2}AQ^{1/2}p - R^{-1/2}y\|_Y\} \\ &= \{w: \|AQ^{1/2}w - y\|_R = \inf_{p \in X} \|AQ^{1/2}p - y\|_R\}. \end{aligned} \tag{3.9}$$

Let v be the element of minimal X-norm in W. Then $v = \tilde{A}^\dagger_{(X,Y)}\tilde{y} = \tilde{A}^\dagger_{(X,Y)}R^{-1/2}y$. On the other hand, upon comparing (3.8) and (3.9) we have $z = Q^{1/2}v = Q^{1/2}\tilde{A}^\dagger_{(X,Y)}R^{-1/2}y$. Thus $z = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{(X,Y)}R^{-1/2}y$ and

$$A^\dagger_{(Q,R)}y = Q^{1/2}(R^{-1/2}AQ^{1/2})^\dagger_{X,Y}R^{-1/2}y,$$

which is the desired result.

COROLLARY 3.1. *If* $A(\mathscr{H}_Q) = \mathscr{H}_R$, *then* $A^{\dagger}_{(Q,R)}$ *is bounded.*

Proof. This follows from Proposition 2.2, or directly from (3.4)–(3.5).

It should be noted that an operator A may satisfy the assumption of Corollary 3.1 while failing to have a closed range in the space Y. This is, for example, the case if A is a Hilbert–Schmidt linear integral operator (with nondegenerate kernel) on X. It is this observation which makes RKHS useful in the context of regularization and approximation of ill-posed linear operator equations. An application of Theorem 3.1 is given in §5.

4. Explicit representations of minimal-norm solutions of linear operator equations in reproducing kernel spaces. We assume that $\mathscr{H}_Q$ is chosen so that

$$\text{the linear functionals } \{\mathscr{E}_t : t \in T\} \text{ defined by } \mathscr{E}_t f = (Af)(t) \text{ are continuous in } \mathscr{H}_Q. \tag{4.1}$$

Then by the Riesz representation theorem, there exists $\{\eta_t, t \in T\} \in \mathscr{H}_Q$ such that

$$(Af)(t) = \langle \eta_t, f \rangle_Q, \qquad t \in T, \quad f \in \mathscr{H}_Q. \tag{4.2}$$

By (2.4), η_t is explicitly given by

$$\eta_t(s) = \langle \eta_t, Q_s \rangle = (AQ_s)(t). \tag{4.3}$$

($\eta_t(s)$ is readily obtained in a more explicit form from (4.3) if A is a differential or integral operator.)

Let $R(t, t')$ be the nonnegative definite kernel on $T \times T$ given by

$$R(t, t') = \langle \eta_t, \eta_{t'} \rangle_Q, \qquad t, t' \in T. \tag{4.4}$$

Let $\mathscr{H}_R$ be the RKHS with RK R given by (4.4). Let R_t be the element of $\mathscr{H}_R$ defined by $R_t(t') = R(t, t')$, and let $\langle \cdot, \cdot \rangle_R$ be the inner product in $\mathscr{H}_R$. Let V be the closure of the span of $\{\eta, t \in T\}$ in $\mathscr{H}_Q$. Now $\{R_t, t \in T\}$ spans $\mathscr{H}_R$, and by the properties of RKHS, we have

$$\langle \eta_t, \eta_{t'} \rangle_Q = R(t, t') = \langle R_t, R_{t'} \rangle_R. \tag{4.5}$$

Thus there is an isometric isomorphism between the subspace V and $\mathscr{H}_R$, generated by the correspondence

$$\eta_t \in V \sim R_t \in \mathscr{H}_R. \tag{4.6}$$

Then $f \in V \sim g \in \mathscr{H}_R$ if and only if $\langle \eta_t, f \rangle_Q = g(t) = \langle R_t, g \rangle_R$, $t \in T$, i.e., if and only if $g(t) = (Af)(t)$, $t \in T$. Thus $A(\mathscr{H}_Q) = A(V) = \mathscr{H}_R$. The nullspace of A in $\mathscr{H}_Q$ is $\{ f : f \in \mathscr{H}_Q, \|Af\|_R = 0\}$. Since

$$\langle \eta_t, f \rangle_Q = 0, \quad t \in T \qquad \text{and} \qquad f \in \mathscr{H}_Q \Rightarrow f \in V^{\perp},$$

and $f \in V$ implies $\|f\|_Q = \|Af\|_V$, it follows that the nullspace of A in $\mathscr{H}_Q$ is $V^{\perp}$ ($\perp$ in $\mathscr{H}_Q$). Hence (4.1) entails that the nullspace of $A : \mathscr{H}_Q \to \mathscr{H}_R$ in $\mathscr{H}_Q$ is always closed, irrespective of the topological properties of $A : X \to Y$.

We list the following table of corresponding sets and elements, under the correspondence $\sim$ of (4.6), where the entries on the left are in $\mathscr{H}_Q$:

$$
\begin{aligned}
V &\sim \mathscr{H}_R, \\
f &\sim g, \\
\eta_t &\sim R_t, \\
P_V Q_s &\sim \eta_s^*.
\end{aligned} \tag{4.7}
$$

Here P_V is the projector from $\mathscr{H}_Q$ onto the (closed) subspace V, $g(t) = \langle \eta_t, f \rangle_Q$, $t \in T$, and $\eta_s^* = AQ_s = A(P_V Q_s)$, i.e.,

$$\eta_s^*(t) = \langle \eta_t, P_V Q_s \rangle_Q = \eta_t(s). \tag{4.8}$$

We have the following theorem.

THEOREM 4.1. *Let A and $\mathscr{H}_Q$ satisfy* (4.1), *and let R be given by* (4.5), *where η_t is defined by* (4.2). *Let $\eta_s^* = AQ_s$. Then, for $g \in \mathscr{H}_R$,*

$$(A^\dagger_{(Q,R)} g)(s) = \langle \eta_S^*, g \rangle_R, \qquad s \in S.$$

Proof. Let $\hat{f}$ be the element in $\mathscr{H}_Q$ of minimal $\mathscr{H}_Q$-norm which satisfies $Af = g$, that is, $\hat{f} = A^\dagger_{(Q,R)} g$. Then $\hat{f} \in V$ and $g \sim \hat{f}$. Also $\eta_s^* \sim P_V Q_s$. Thus

$$\hat{f}(s) = \langle Q_s, \hat{f} \rangle_Q = \langle P_V Q_s, \hat{f} \rangle_Q = \langle \eta_s^*, g \rangle_R.$$

We next obtain another operator representation of $A^\dagger_{(Q,R)}$.

THEOREM 4.2. *Suppose*

(i) *$\mathscr{D}(A^*)$ is dense in Y, where A^* is the adjoint of A considered as an operator from X to Y;*

(ii) *A and $\mathscr{H}_Q$ satisfy* (4.1);

(iii) *$\mathscr{H}_Q$ and $\mathscr{H}_R = A(\mathscr{H}_Q)$ possess continuous RK's.*

Then, for $g \in \mathscr{H}_R$,

$$(A^\dagger_{(Q,R)} g)(s) = (QA^*(AQA^*)^\dagger_{(Y,Y)} g)(s), \qquad s \in S.$$

Proof. First we show that $R = AQA^*$. This follows by observing that, for $g \in \mathscr{D}(A^*)$, (4.2), (2.12), (4.7) and the isomorphism between V and $\mathscr{H}_R$ give

$$
\begin{aligned}
(AQA^* g)(t) = \langle \eta_t, QA^* g \rangle_Q &= (\eta_t, A^* g)_X \\
&= (A\eta_t, g)_Y = (R_t, g)_Y \\
&= \int_T R(t, t') g(t')\, dt', \qquad t \in T.
\end{aligned}
$$

Thus, AQA^* coincides with the bounded operator R on $\mathscr{D}(A^*)$ and hence by extension on Y. We write $(AQA^*)^\dagger_{(Y,Y)} = R^{-1}$. Next, suppose $g \in R(\mathscr{D}(A^*))$, and let $\rho = R^{-1} g$. Then, since $g = R\rho$, Theorem 4.1 and (2.12) give

$$
\begin{aligned}
(A^\dagger_{(Q,R)} g)(s) &= \langle \eta_s^*, g \rangle_R = (\eta_s^*, \rho)_Y = (AQ_s, \rho)_Y \\
&= (Q_s, A^*\rho)_X = (QA^*\rho)(s) = (QA^*(AQA^*)^\dagger_{(Y,Y)} g)(s), \qquad s \in S.
\end{aligned}
$$

It can be shown easily that if $\mathscr{D}(A^*)$ is dense in Y, then $R(\mathscr{D}(A^*))$ is dense in $\mathscr{H}_R$. Thus (4.9) extends to all $g \in \mathscr{H}_R$.

DEFINITION 4.1. Let $A:X \to Y$. The *pseudocondition number* of A (relative to the norms of X and Y) is

$$\gamma(A;X,Y) := \sup_{\substack{x\neq 0\\ x\in\mathscr{D}(A)}} \frac{\|Ax\|_Y}{\|x\|_X} \cdot \sup_{\substack{y\neq 0\\ y\in\mathscr{D}(A^\dagger)}} \frac{\|A^\dagger y\|_X}{\|y\|_Y}.$$

The equation $Af = g$ is said to be *poorly conditioned* in the spaces X, Y if the number $\gamma(A;X,Y)$ is much greater than one. Note that $1 \leqq \gamma(A;X,Y)$; for ill-posed problems, γ is not finite.

Suppose $\mathscr{H}_Q$ is an RKHS with $\mathscr{H}_Q \subset \mathscr{D}(A)$, and A and $\mathscr{H}_Q$ satisfy (4.1) with $A(\mathscr{H}_Q) = \mathscr{H}_R$, R given by (4.4). Then $\gamma(A;\mathscr{H}_Q,\mathscr{H}_R) = 1$. To see this, write $x \in \mathscr{H}_Q$ in the form $x = x_1 + x_2$, where $x_2 \in V^\perp$. Then $Ax = Ax_1 = y_1$ and $\|y_1\|_R = \|x_1\|_Q$. Thus

$$\gamma(A;\mathscr{H}_Q,\mathscr{H}_R) = \sup_{x\neq 0} \frac{\|y_1\|_R}{\|x\|_Q} \cdot \sup_{y_1\neq 0} \frac{\|x_1\|_Q}{\|y_1\|_R} = 1.$$

On the other hand, the number $\gamma(A;X,Y)$ may be large. Thus the casting of the operator equation $Af = g$ in the reproducing kernel spaces $\mathscr{H}_Q$, $\mathscr{H}_R$ always leads to a well-conditioned (indeed, optimally-conditioned) problem.

5. Regularization of pseudosolutions in reproducing kernel spaces. In this section we study properties of *regularized pseudosolutions* (in RKHS) f_λ of the operator equation $Af = g$, where g is not necessarily in the range of the operator A. By a regularized pseudosolution we mean a solution to the variational problem: Find f_λ in $\mathscr{H}_Q$ to minimize

$$\phi_g(f) = \|g - Af\|_P^2 + \lambda\|f\|_Q^2, \tag{5.1}$$

where $\mathscr{H}_Q$ is an RKHS in the domain of A, $\|\cdot\|_P$ denotes the norm in an RKHS $\mathscr{H}_P$ with RK P, $\mathscr{H}_P \subset Y$, $\phi_g(f)$ is assigned the value $+\infty$ if $g - Af \notin \mathscr{H}_P$, and $\lambda > 0$. We suppose A and $\mathscr{H}_Q$ satisfy (4.1), hence $A(\mathscr{H}_Q) = \mathscr{H}_R$, where $\mathscr{H}_R$ possesses an RK. As before, A may be unbounded, invertible, or compact considered as an operator from $X(=\mathscr{L}_2[S])$ to $Y(=\mathscr{L}_2[T])$. It is assumed that g possesses a (not necessarily unique) representation $g = g_0 + \xi$, for some $g_0 \in A(\mathscr{H}_Q)$ and $\xi \in \mathscr{H}_P$. ξ may be thought of as a "disturbance."

For $\lambda > 0$, let $\mathscr{H}_{\lambda P}$ be the RKHS with RK $\lambda P(t,t')$, where $P(t,t')$ is the RK on $T \times T$ associated with $\mathscr{H}_p$. We have $\mathscr{H}_P = \mathscr{H}_{\lambda P}$ and

$$\|\cdot\|_P^2 = \lambda\|\cdot\|_{\lambda P}^2. \tag{5.2}$$

Let $R(\lambda) = R + \lambda P$, and let $\mathscr{H}_{R(\lambda)}$ be the RKHS with RK $R(\lambda) = R(\lambda;t,t')$. According to Aronszajn [1, p. 352], $\mathscr{H}_{R(\lambda)}$ is the Hilbert space of functions of the form

$$g = g_0 + \xi, \tag{5.3}$$

where $g_0 \in \mathscr{H}_R$ and $\xi \in \mathscr{H}_P$. Following Aronszajn [1], we note that this decomposition is *not* unique unless $\mathscr{H}_R$ and $\mathscr{H}_P$ have no element in common except the zero element. The norm in $\mathscr{H}_{R(\lambda)}$ is given by

$$\|g\|_{R(\lambda)}^2 = \min\{\|g_0\|_R^2 + \|\xi\|_{\lambda P}^2 : g_0 \in \mathscr{H}_R, \xi \in \mathscr{H}_P, g_0 + \xi = g\}, \tag{5.4}$$

where, however, the g_0 and ξ attaining the minimum in (5.4) are easily shown to be unique by the strict convexity of the norm.

Consider now the problem of finding $f_\lambda \in \mathscr{H}_Q$ to minimize $\phi_g(f)$ in (5.1), for $g \in \mathscr{H}_{R(\lambda)}$. Then $g - Af_\lambda$ must be in $\mathscr{H}_P$ and it is obvious that $f_\lambda \in V$, the orthogonal complement of the nullspace of A in $\mathscr{H}_Q$. For any $f \in V$, $\|f\|_Q = \|Af\|_R$ by the isometric isomorphism between V and $\mathscr{H}_R$, and (5.1) may be written in the equivalent form: Find $f_\lambda \in V$ to minimize

$$\lambda\|Af\|_R^2 + \|g - Af\|_P^2. \tag{5.5}$$

Comparing (5.4) and (5.5) with the aid of (5.2), we see that g_0 and ξ attaining the minimum on the right-hand side of (5.4) are related to the solution f_λ, of the minimization problem (5.5), by

$$g_0 = Af_\lambda \quad \text{and} \quad \xi = g - Af_\lambda.$$

In the following theorem, we give a representation of the solution f_λ.

THEOREM 5.1. *Suppose $\mathscr{D}(A^*)$ is dense in Y, $\mathscr{H}_Q \subset \mathscr{D}(A)$ and A and $\mathscr{H}_Q$ satisfy* (4.1). *Suppose $\mathscr{H}_Q$, $\mathscr{H}_R(=A(\mathscr{H}_Q))$ and $\mathscr{H}_P \subset Y$ all have continuous RK's. Then, for $g \in \mathscr{H}_{R(\lambda)}$, the unique minimizing element $f_\lambda \in \mathscr{H}_Q$ of the functional $\phi_g(f)$ is given by*

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = f_\lambda(s) = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g)(s), \qquad s \in S, \tag{5.6}$$

where $\eta_s^* = AQ_s$.

Proof. First, our assumptions give that $AQA^* + \lambda P(=R + \lambda P)$ is a well-defined positive definite operator on Y. We demonstrate, for

$$g \in (AQA^* + \lambda P)(\mathscr{D}(A^*)),$$

that

$$f_\lambda = QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g. \tag{5.7}$$

Now, $g - Af_\lambda = \lambda P(AQA^* + \lambda P)^\dagger_{(Y,Y)}g \in \mathscr{H}_P$, so that this demonstration will be effected if we show that

$$\phi_g(f_\lambda) < \phi_g(f_\lambda + \delta)$$

for any $\delta \in \mathscr{H}_Q$, with $\|\delta\|_Q \neq 0$.

But

$$\begin{aligned}
\phi_g(f_\lambda + \delta) &= \|\lambda P(AQA^* + \lambda P)^\dagger_{(Y,Y)}g\|_P^2 - 2\lambda((AQA^* + \lambda P)^\dagger_{(Y,Y)}g, A\delta)_Y + \|A\delta\|_P^2 \\
&\quad + \lambda\|QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g\|_Q^2 + 2\lambda(A^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g, \delta)_X \\
&\quad + \|\delta\|_Q^2 \\
&= \phi_g(f_\lambda) + \|A\delta\|_P^2 + \|\delta\|_Q^2 > \phi_g(f_\lambda), \qquad \delta \neq 0.
\end{aligned}$$

We next show that, for $g \in (AQA^* + \lambda P)(\mathscr{D}(A^*))$, that

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g)(s).$$

Let $(AQA^* + \lambda P)^\dagger g = \rho \in \mathscr{D}(A^*)$. Then using (2.12) with Q replaced by $R(\lambda)$ gives

$$\langle \eta_s^*, g\rangle_{R(\lambda)} = (\eta_s^*, \rho)_Y = (AQ_s, \rho)_Y = (Q_s, A^*\rho)_X$$
$$= (QA^*\rho)(s) = (QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)}g)(s).$$

Thus we have proved (5.6) for $g \in (AQA^* + \lambda P)(\mathscr{D}(A^*))$.

We next show that $QA^*(AQA^* + \lambda P)^\dagger_{(Y,Y)} \equiv QA^*(R + \lambda P)^{-1}$ defines a bounded linear operator from $\mathscr{H}_{R(\lambda)}$ to $\mathscr{H}_Q$. If $g \in \mathscr{H}_{R(\lambda)}$, then $(R + \lambda P)^{-1/2}g \Rightarrow p \in Y \ominus \mathscr{N}(R + \lambda P)$ and

$$Q^{1/2}A^*(R + \lambda P)^{-1}g = Q^{1/2}A^*(R + \lambda P)^{-1/2}p \in Y,$$

since

$$\|Q^{1/2}A^*(R + \lambda P)^{-1/2}p\|_Y = \|R^{1/2}(R + \lambda P)^{-1/2}p\|_Y \leqq \|p\|_Y.$$

Therefore

$$QA^*(R + \lambda P)^{-1}g \in \mathscr{H}_Q, \qquad g \in \mathscr{H}_{R(\lambda)}.$$

But

$$\|QA^*(R + \lambda P)^{-1}g\|_Q = Q^{1/2}A^*(R + \lambda P)^{-1/2}p\|_Y \leqq \|p\|_Y = \|g\|_{R(\lambda)}.$$

It can be shown that $(R + \lambda P)(\mathscr{D}(A^*))$ is dense in $\mathscr{H}_{R(\lambda)}$, so that the right-hand equality in (5.6) extends to all $g \in \mathscr{H}_{R(\lambda)}$, and the left-hand equality obviously extends by the continuity of the inner product.

We call the (linear) mapping which assigns (by Theorem 5.1) to each $g \in \mathscr{H}_{R(\lambda)}$ the unique minimizing element f_λ the *regularization operator* of the equation $Af = g$.

The most useful situations occur, of course, when $\mathscr{H}_R$ is strictly contained in $\mathscr{H}_{R(\lambda)}$. For example, $\mathscr{H}_R$ may be a dense subset of Y in the Y-topology and $\mathscr{H}_{R(\lambda)}$ a bigger dense subset. We discuss this case further in §6. On the other hand, if $\mathscr{H}_R^\perp$ (in Y) is not empty, then P may be chosen so that the closure of $\mathscr{H}_P$ in the Y-topology equals $\mathscr{H}_R^\perp$ in Y. Then $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, $\mathscr{H}_{\lambda P}$ and $\mathscr{H}_R$ are orthogonal subspaces of $\mathscr{H}_{R(\lambda)}$ (see [1]), and the decomposition (5.3) is unique. In this case we have the following theorem which shows that the regularization operator is indeed a generalized inverse in an appropriate RKHS.

THEOREM 5.2. *If $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, then the minimizing element f_λ of (5.1) is the solution to the problem: Find $f \in \mathscr{S}$ to minimize*

$$\|f\|_Q, \tag{5.8}$$

where

$$\mathscr{S} = \{f : f \in \mathscr{H}_Q, \|g - Af\|_{R(\lambda)} = \inf_{h \in \mathscr{H}_Q} \|g - Ah\|_{R(\lambda)}\}. \tag{5.9}$$

Proof. We first note that if $\mathscr{H}_P \cap \mathscr{H}_R = \{0\}$, then also $\mathscr{H}_{\lambda P} \cap \mathscr{H}_R = \{0\}$ and the decomposition $g = g_0 + \xi$ with $g_0 \in \mathscr{H}_R$ and $\xi \in \mathscr{H}_{\lambda P}$ is unique, with

$$g_0 = R(R + \lambda P)^{-1}g \quad \text{and} \quad \xi = \lambda P(R + \lambda P)^{-1}g.$$

This decomposition is also independent of λ in this case, $PR = RP = 0$, and $R(R + \lambda P)^{-1}$ is the restriction of the projection onto $\mathscr{N}(R)^\perp$ in Y to the domain

$R^{1/2}Y \oplus P^{1/2}Y$. We have

$$\|g - Af\|_{R(\lambda)}^2 = \|g_0 + \xi - Af\|_{R(\lambda)}^2 = \|g_0 - Af\|_R^2 + \|\xi\|_{\lambda R}^2.$$

Thus since $g_0 \in A(\mathscr{H}_Q)$, $\inf\{\|g - Af\|_{R(\lambda)} : f \in \mathscr{H}_Q\} = \|\xi\|_{\lambda P}$ and $\mathscr{S} = \{f : f \in \mathscr{H}_Q, Af = g_0\}$. Hence $f_\lambda = A^\dagger_{(Q,R)} g_0 = QA^*R^{-1}g_0 = QA^*(R + \lambda P)^{-1}g$.

Remark 5.1. In our setting we have

$$A(\mathscr{H}_Q) = \mathscr{H}_R \subset \mathscr{H}_{R(\lambda)} \subset Y.$$

Replacing $\mathscr{H}_{\tilde{R}}$ and $\mathscr{H}_R$ in (3.2) by $\mathscr{H}_R$ and $\mathscr{H}_{R(\lambda)}$, respectively, we get from (3.5):

$$A^\dagger_{(Q,R(\lambda))} y = Q^{1/2}[(R + \lambda P)^{-1/2} A Q^{1/2}]^\dagger_{(X,Y)} (R + \lambda P)^{-1/2} y \tag{5.10}$$

for $y \in \mathscr{D}(A^\dagger_{(Q,R(\lambda)})$; see (3.4.).

It is helpful to remember that the topology on $\mathscr{H}_R$ is not, in general, the restriction of the topology of $\mathscr{H}_{R(\lambda)}$, with the notable exception of the case $\mathscr{H}_R \cap \mathscr{H}_P = \{0\}$. In [11] the authors provide a concrete example arising in the approximate solution of boundary value problems where $\mathscr{H}_R$ is not a closed subspace of $\mathscr{H}_R$.

If $\mathscr{H}_R \cap \mathscr{H}_P = \{0\}$, then $\mathscr{H}_R$ is a closed subspace of $\mathscr{H}_{R(\lambda)}$ and (by Theorem 5.2)

$$A^\dagger_{(Q,R(\lambda))} = QA^*(R + \lambda P)^{-1}. \tag{5.11}$$

Note that in this case, the generalized inverse and the regularization operator coincide.

If $\mathscr{H}_R = A(\mathscr{H}_Q)$ is not closed in $\mathscr{H}_{R(\lambda)}$, then the regularization operator and the generalized inverse are different. Also, the right-hand sides of (5.10) and (5.11) are not the same: (5.11) has maximal domain $\mathscr{H}_{R(\lambda)}$, while (5.10) has maximal domain $\mathscr{H}_R \oplus \mathscr{H}_R^\perp$ ($\perp$ in $\mathscr{H}_{R(\lambda)}$).

6. Properties of f_λ when $\mathscr{H}_R \subset \mathscr{H}_P$. Rates of convergence of f_λ to the generalized inverse. In this section we note some properties of f_λ as $\lambda \to 0$ when $\mathscr{H}_R \subset \mathscr{H}_P$. If $g \in \mathscr{H}_R = A(\mathscr{H}_Q)$, then we have $f_\lambda \to A^\dagger_{(Q,R)}g$ as $\lambda \to 0$; here we may say something about the rate of convergence if certain additional conditions are satisfied (compare also with Ivanov and Kudrinskii [4]). However, g may not be in the domain of $A^\dagger_{(Q,R)}$. This situation can occur if, for example, $\mathscr{H}_R$ is dense in $\mathscr{H}_{R(1)}$. In this case, $\lim_{\lambda \to 0} \|f_\lambda\|_Q = \infty$.

THEOREM 6.1. *Let $g = Af_0 + \xi_0$, where $f_0 \in V$, $\xi_0 \in \mathscr{H}_P$, and suppose that $\mathscr{H}_R \subset \mathscr{H}_P$. Then*

(i) *$B = P^{-1/2}R^{1/2}$ is a bounded operator on $Y = \mathscr{L}_2[T]$;*

(ii) *if $\xi_0 = 0$ and $\|(B^*B)^{-1}R^{-1/2}(Af_0)\|_{\mathscr{L}_2[T]} < \infty$, then*

$$\|A^\dagger_{(Q,R)}g - f_\lambda\|_Q^2 = O(\lambda^2);$$

(iii) *if $\xi_0 = 0$ and $\|(B^*B)^{-1/2}R^{-1/2}(Af_0)\|_{\mathscr{L}_2[T]} < \infty$, then*

$$\|A^\dagger_{(Q,R)}g - f_\lambda\|_Q^2 = O(\lambda);$$

(iv) *if $\xi_0 \notin \mathscr{H}_R$, then $\lim_{\lambda \to 0} \|f_\lambda\|_Q = \infty$.*

Here inverses indicated by $^-$ are the generalized inverses in the geometry of $\mathscr{L}_2$-spaces.

Proof. Assertion (i) follows from the fact that $\mathscr{H}_R = R^{1/2}(\mathscr{L}_2[T])$ and $\mathscr{H}_P = P^{1/2}(\mathscr{L}_2[T])$. If $\mathscr{H}_R \subset \mathscr{H}_P$, then $R^{1/2}(\mathscr{L}_2[T]) \subset P^{1/2}(\mathscr{L}_2[T])$, so that $P^{-1/2}R^{1/2}$ is bounded. To prove assertions (ii) and (iii), we note that since $A(\mathscr{H}_Q) = \mathscr{H}_R$, $R^{-1/2}(Af_0)$ is a well-defined element ϕ of $\mathscr{L}_2[T]$, and after some computation, we obtain that if $\xi_0 = 0$, then

$$\begin{aligned}\|A^\dagger_{(Q,R)}g - f_\lambda\|_R &= \|(I - R^{1/2}(R + \lambda P)^{-1}R^{1/2})\phi\|_{\mathscr{L}_2[T]} \\ &= \|\lambda(B^*B + \lambda I)^{-1}\phi\|_{\mathscr{L}_2[T]} \\ &= \lambda\|(B^*B + \lambda I)^{-1}\phi\|_{\mathscr{L}_2[T]} \\ &\leqq \lambda\|(B^*B)^{-1}\phi\|_{\mathscr{L}_2[T]}.\end{aligned}$$

If $\|(B^*B)^{-1}R^{-1/2}Af_0)\|_{\mathscr{L}_2[T]} = m < \infty$, then

$$\|A^\dagger_{(Q,R)}g - f_\lambda\|_R \leq \lambda m,$$

thus proving assertion (ii).

Assertion (iii) follows by noting that $\lambda(B^*B + \lambda I)^{-1} \leqq I$ in the sense of positive definiteness; thus $\lambda(B^*B + \lambda I)^{-1} \leqq \lambda^{1/2}(B^*B + \lambda I)^{-1/2}$. Hence,

$$\begin{aligned}\lambda^2\|B^*B + \lambda I)^{-1}\phi\|^2_{\mathscr{L}_2[T]} &\leqq \lambda\|(B^*B + \lambda I)^{-1/2}\phi\|^2_{\mathscr{L}_2[T]} \\ &\leqq \lambda\|(B^*B)^{-1/2}\phi\|^2_{\mathscr{L}_2[T]},\end{aligned}$$

giving assertion (iii).

To see (iv), we observe that

$$\|QA^*(AQA^* + \lambda P)^{-1}\xi\|_Q = \|Q^{1/2}A^*(AQA^* + \lambda P)^{-1}\xi\|_{\mathscr{L}_2[S]}.$$

Since $\xi \in \mathscr{H}_P$, we have $\xi = P^{1/2}\theta$ for some $\theta \in \eta(P)^\perp$ ($\perp$ in $\mathscr{L}_2[T]$). Then

$$\|Q^{1/2}A^*(AQA^* + \lambda P)^{-1}P^{1/2}\theta\|_{\mathscr{L}_2[S]} = \|(BB^*)^{1/2}(BB^* + \lambda I)^{-1}\theta\|_{\mathscr{L}_2[S]}.$$

If $\{\lambda_\nu, \phi_\nu\}_{\nu=1}^\infty$ are the eigenvalues and eigenfunctions of the bounded positive operator BB^*, then

$$\|(BB^*)^{1/2}(BB^* + \lambda I)^{-1}\theta\|^2_{\mathscr{L}_2[S]} = \sum_{\nu=1}^\infty \frac{\lambda_\nu}{(\lambda_\nu + \lambda)^2}(\phi_\nu, \theta)^2_{\mathscr{L}_2[S]}.$$

Since $\xi \notin \mathscr{H}_R$, $P^{1/2}\theta$ is not in the domain of $R^{-1/2}$ and θ is not in the domain of B^{-1}. Thus

$$\|(BB^*)^{-1/2}\theta\|^2_{\mathscr{L}_2[S]} = \sum_{\nu=1}^\infty \frac{1}{\lambda_\nu}(\phi_\nu, \theta)^2_{\mathscr{L}_2[S]} = \infty$$

and

$$\lim_{\lambda \to 0}\|(BB^*)(BB^* + \lambda I)^{-1}\theta\|_{\mathscr{L}_2[S]} = \infty.$$

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