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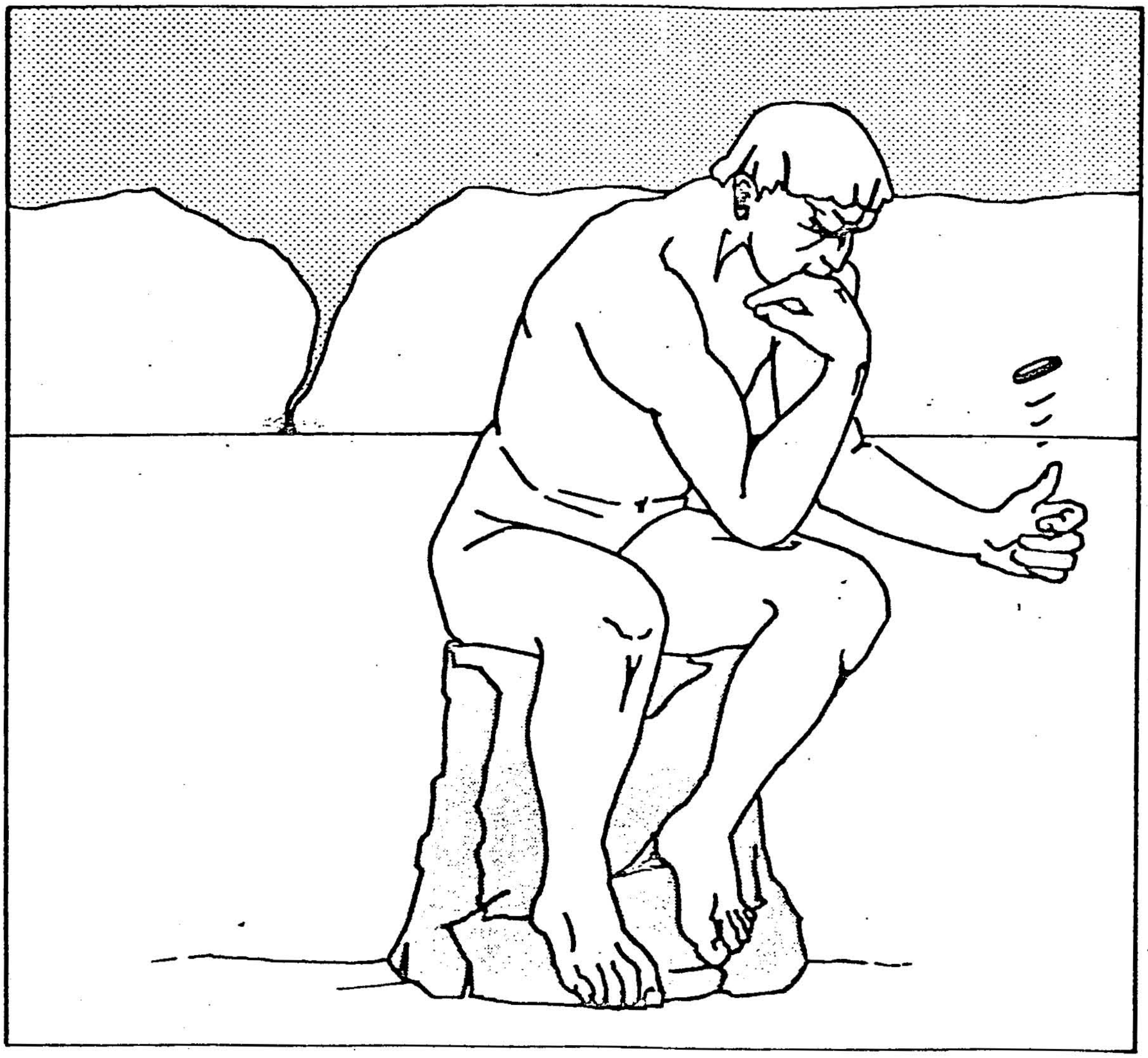

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SOME NEW TECHNIQUES FOR VARIATIONAL OBJECTIVE ANALYSIS ON THE SPHERE USING SPLINES, HOUGH FUNCTIONS, AND SAMPLE SPECTRAL DATA

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1. INTRODUCTION

The purpose of this work is to develop a new class of variational objective analysis methods on the sphere. This work extends the work of Wahba and Wendelberger (1980) on the plane and in three dimensions, to the sphere. This class of methods is suitable for the simultaneous analysis of both conventional and satellite data although we do not discuss those details here. The methods described have the potential of incorporating in a global analysis, realistic anisotropy, as discussed by Ghil, Balgovind and Kalnay-Rivas (1981) and Thiebaux (1977). A parsimonious parametrization (Thiebaux (1981)) which reflects observed or theoretically derived decrease of the spectral distribution of energy with wave number, is incorporated, and dynamic estimation of a small number of carefully selected free parameters is made from the data being analyzed.

The methods have the feature that the resulting fields can be differentiated and integrated analytically, providing a tool for estimating vorticity, divergence, transport processes, budgets, etc. In this paper we discuss the analysis of a set of observational data, and the development of specific models from historical observations on spectral components. In numerical weather prediction models, the objective analysis is typically carried out on a difference field, observations minus predictions, where the predicted information is obtained from a forecast corresponding to the observation time. The analyzed (observed minus predicted) field is then added to the predicted field to obtain new initial conditions. The analysis will not in general be the same as if only data are being analyzed (Jones (1965) and Ghil et al (1980) are relevant here). However, we believe that the class of models proposed here is likely to be appropriate for the analysis of observed minus predicted fields. The details of the optimum models have to be established in conjunction with a particular forecast model however, as they will be dependent on the dynamics of the forecast model, and further work remains to be done.

2. ISOTROPIC SPLINE SMOOTHING ON THE SPHERE

First, the isotropic variational objective analysis methods in Wahba and Wendelberger (1980) for data observed on a plane, are extended to data on one variable observed on the sphere. Let the data be modelled as

$$z_i = f(P_i) + \varepsilon_i, \quad i=1,2,\ldots,N \tag{1}$$

where P_i is a point on the sphere, $f(P_i)$ is the "true" value of the quantity of interest at P_i (the 500 mb. ht., for example) and the ε_i are errors modelled as independent zero mean random variables with common unknown variance σ^2. Given $z_1,\ldots,z_N$, f is estimated as the minimizer, call it $f_{m,\lambda}$, (in an appropriate function space) of

$$\frac{1}{N}\sum_{i=1}^{N}(z_i-f(P_i))^2 + \lambda J_m(f) \tag{2}$$

where

$$J_m(f) = \int_0^{2\pi}\int_0^{\pi}(\Delta^{m/2}f)^2(\theta,\phi)\sin\theta\, d\theta\, d\phi, \quad m \text{ even}$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{(\Delta^{(m-1)/2}f)_\phi^2}{\sin\theta^2}$$

$$+ (\Delta^{(m-1)/2}f)_\theta^2 \sin\theta\, d\theta\, d\phi, \quad m \text{ odd},$$

and Δ is the Laplacian on the sphere,

$$\Delta f = \frac{1}{\sin^2\theta} f_{\phi\phi} + \frac{1}{\sin\theta}(\sin\theta f_\theta)_\theta,$$

θ = colatitude, ϕ = longitude, $(\theta,\phi) = P$. The estimate $f_{m,\lambda}$ may be considered to be a generalization to the sphere of the notion of a polynomial spline on the real line. It may also be considered to be a low pass (Butterworth) filtering of the data with respect to the spherical harmonics. λ controls the half power point and m the steepness of the roll off of the filter. In any method of this type where the data are noisy the results can be expected to be sensitive to the choice of λ. (λ can also be considered as a proxy for the effective signal to noise ratio, see below).

The parameters λ and m can be preset or can be estimated dynamically from the data being analyzed by generalized cross validation. As in Wahba and Wendelberger (1980), and Wendelberger (1981), data in the form of integrals or derivatives of f can be objectively analyzed simultaneously with data on $\{f(P_i)\}$, and estimates of integrals or derivatives of f can be made by integrating or differentiating $f_{m,\lambda}$.

Wendelberger (1981) has developed some fast numerical methods which can be used to compute $f_{m,\lambda}$ and estimate m and λ by generalized cross validation, which appear to work efficiently for up to around 200-300 data points. Using methods suggested in Utreras (1979) and Wahba (1980) it is believed that Wendelberger's numerical methods can be efficiently extended to the simultaneous analysis of several times that many data points.

3. VARIATIONAL METHODS, LANGEVIN EQUATIONS, AND GANDIN METHODS

There are intimate relations between certain forms of variational objective analyses, Langevin equation models and Gandin (or Bayes) objective analysis. See Kimeldorf and Wahba (1970), Wahba (1978). By smoothing the data using $f_{m,\lambda}$ it can be shown that one will get the same estimate for f (up to but not including the choice of λ), as if one modelled f as a random function satisfying the Langevin equation

$$\Delta^{m/2} f = \text{"white noise"} \quad (m \text{ even}). \tag{3}$$

Through this observation our work is related to that of Ghil, Balgovind and Kalnay-Rivas (1981). It can be shown that the model (3) leads to the same result as modelling f as a random combination of spherical harmonics:

$$f(P) = \bar{f} + \sum_{\substack{n,s \\ n\neq 0}} \xi_{ns} Y_{ns}(P) \tag{4}$$

where $\bar{f}$ is a constant, the Y_{ns}, $|s|\leq n$ are normalized spherical harmonics and the ξ_{ns} are independent, zero mean normally distributed random variables with

$$E\xi^2_{ns} = b\lambda_{ns}, \tag{5}$$

b is an unknown constant and $\lambda_{ns} = [n(n+1)]^{-m}$. σ^2/Nb plays the role of λ. (Equation (4) is also called the Karhunen-Loeve expansion).

The proof that (3) and (4) are the same model follows from the fact that λ^{-1}_{ns} and Y_{ns} are the eigenvalues and eigenfunctions of $\Delta^{m/2}(\Delta^{m/2})^*$. (For further details see Wahba (1979)). The sequence $\{b\lambda_{ns}\}$ should be viewed as the spectral distribution of f, and hence m governs the rate of decay of the energy spectrum with wave number. In practice m is not required to be an integer. For later reference, we observe that if

$$f(P) = \bar{f} + \sum_{\substack{n,s \\ n\neq 0}} f_{ns} Y_{ns}(P) \tag{6}$$

then

$$J_m(f) = \sum_{\substack{n,s \\ n\neq 0}} f^2_{ns}/\lambda_{ns} . \tag{7}$$

$f_{m,\lambda}$ can be related to a Gandin (or Bayes) estimate of f by deriving the covariance $R(P,P') = Ef(P)f(P')$ from (4), it is

$$R(P,P') = c + b \sum_{\substack{n,s \\ n\neq 0}} \lambda_{ns} Y_{ns}(P) Y_{ns}(P') \tag{8}$$

where c is some constant. Assuming the random model for the data

$$z_i = f(P_i) + \epsilon_i, \; i=1,2,\ldots,N$$

with f having the prior covariance (8) and letting $c \to \infty$, it can be shown that $f_{m,\lambda}(P)$ is the conditional expectation of f(P) given $z_1,\ldots,z_N$, if $\lambda = \sigma^2/Nb$. (Taking $c \to \infty$ here essentially means that the mean of f is completely determined by that data). Proofs are in Kimeldorf and Wahba (1970) and Wahba (1978).

4. OTHER ISOTROPIC MODELS

The spectral sequence $\{b\lambda_{ns}\}$ of (5) and the Langevin equation $\Delta^{m/2}f$ = "white noise" of (3) can be generalized by replacing $\Delta^{m/2}$ by $L_{m/2}$ defined by

$$L_{m/2} = \sum_{j=0}^{m/2} \alpha_j \Delta^j,$$

then the spectral sequence $\{\lambda_{ns}\}$ is given by

$$\lambda_{ns} = \Big(\sum_{j=0}^{m/2} \alpha_j [-n(n+1)]^j\Big)^2 , \tag{9}$$

thus the shape of the spectral sequence $\{b\lambda_{ns}\}$ can be controlled by the choice of the $\{\alpha_j\}$. However only a small number of the α_j should be chosen dynamically (for further discussion of this point, see Wahba (1981)).

5. ANISOTROPIC MODELS

If isotropic models are adequate, then data similar to that collected by Stanford (1979) can be used to develop a model for the λ_{ns} in (8) by, e.g. fitting a model of the form (9) to the data similar to that in his Table 1. The work of Stanford (1979), Ghil, Balgovind and Kalnay-Rivas (1981), and Thiebaux (1977), however, suggests that anisotropic models for some meteorological fields are more realistic. Ghil, Balgovind and Kalnay-Rivas (1981) suggest the use of a Langevin equation based on the potential vorticity equation (instead of (3)), and show that realistic anisotropic covariance functions can be obtained this way (Ghil, personal communication).

We now suggest a related variational approach to obtaining realistic anisotropic covariance functions for objective analysis of mass and wind fields. Let

$$\underset{\sim}{H}_{ns}(P) = \begin{pmatrix} U_{ns}(P) \\ V_{ns}(P) \\ H_{ns}(P) \end{pmatrix}$$

be the Hough harmonics for an appropriate equivalent height. ($U_{ns}(P) = \cos s\phi\ \hat{U}_n^s(\theta)$, $\sin s\phi\ \hat{U}_n^s(\theta)$, etc. of Kasahara, (1976), equations (6.2)-(6.4)). The solutions to Laplace's tidal equations on the sphere, for a fixed time, are then of the form

$$\underset{\sim}{H}(P) = \begin{pmatrix} U(P) \\ V(P) \\ H(P) \end{pmatrix} = \sum_{n,s} c_{ns} \underset{\sim}{H}_{ns}(P),$$

where the c_{ns} are constants. We now model the height and wind field $\underset{\sim}{H}(P)$ as

$$\underset{\sim}{H}(P) = \begin{pmatrix} \Sigma\, \xi_{ns}^U U_{ns}(P) \\ \Sigma\, \xi_{ns}^V V_{ns}(P) \\ \Sigma\, \xi_{ns}^H H_{ns}(P) \end{pmatrix}$$

where the ξ_{ns} are zero mean normally distributed random variables with

$$E(\xi_{ns}^U)^2 = E(\xi_{ns}^V)^2 = E(\xi_{ns}^H)^2 = b\lambda_{ns}\ .$$

It appears that data such as that collected by Kasahara (1976) and Kasahara and Puri (1981) can in principle be used to construct reasonable models for the shape of the sequence $\{\lambda_{ns}\}$. Approximate balance can be enforced by modelling a high degree of correlation between ξ_{ns}^U, ξ_{ns}^V and ξ_{ns}^H for the same ns. For notational simplicity set correlation $(\xi_{ns}^U, \xi_{ns}^V) = \mathrm{corr}(\xi_{ns}^U, \xi_{ns}^H) = \mathrm{corr}(\xi_{ns}^V, \xi_{ns}^H) = \rho$, and let all the other correlation coefficients be 0. Leting u_i, v_i and h_i be observations

$$\begin{aligned} u_i &= U(P_i) + \epsilon_i^U \\ v_i &= V(P_i) + \epsilon_i^V \qquad i=1,2,\ldots,N, \\ h_i &= H(P_i) + \epsilon_i^H \end{aligned}$$

then this model leads to a variational objective analysis of the form: Find $\underset{\sim}{H}_\lambda$ (in an appropriate space of functions) to minimize

$$\frac{1}{N}\sum_{i=1}^{N}(U(P_i)-u_i)^2 + \frac{1}{N}\sum_{i=1}^{N}(V(P_i)-v_i)^2 + \frac{W}{N}\sum_{i=1}^{N}(H(P_i)-h_i)^2 + \lambda J(\underset{\sim}{H}),$$

where

$$J(\underset{\sim}{H}) = (1+\rho)(\langle U,U\rangle + \langle V,V\rangle + \langle H,H\rangle) - 2\rho(\langle U,V\rangle + \langle U,H\rangle + \langle V,H\rangle), \qquad (10)$$

with

$$\langle U,V\rangle = \sum_{ns} \frac{\tilde{U}_{ns}\tilde{V}_{ns}}{\lambda_{ns}}$$

where

$$\tilde{U}_{ns} = \int_S U \cdot U_{ns}(P)dP$$

$$\tilde{V}_{ns} = \int_S V \cdot V_{ns}(P)dP,$$

and the other terms are defined analogously. W represents the relative accuracy in measuring winds and heights. Using the theory of reproducing kernels (see e.g. Wahba (1973, 1978) explicit expressions for the minimizer of (10) can be obtained, and it is believed that feasible numerical algorithms can also be obtained. Gravity waves are partially suppressed by setting the λ_{ns} corresponding to gravity waves small and completely suppressed by eliminating them from the solution space. Three dimensional generalizations using vertical structure functions or EOF's can also be developed.

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