Statistics 860 Lecture 17

Splines on the Sphere, Diffusion Covariances on the Sphere, Other Isotropic Covariances on the Sphere

- G. Wahba. Spline interpolation and smoothing on the sphere. *SIAM J. Sci. Stat. Comput.*, 2:5–16, 1981. `sphspl.pdf`.

- G. Wahba. Erratum: Spline interpolation and smoothing on the sphere. *SIAM J. Sci. Stat. Comput.*, 3:385–386, 1982.

- A. Weaver and P. Courtier. Correlation modelling on the sphere using a generalized diffusion equation. *Q. J. R. Meterol. Soc*, 127:1815–1846, 2001.

Overhead slides for talk at the IMS Mini-meeting on Statistical Approaches to the Ocean Circulation Inverse Problem, 2001. See TALKS link.

Spherical Harmonics References:

1. E. T. Whittaker and G. M. Watson, A course of modern analysis.

2. M. Abromowitz and I. Stegun, Handbook of Mathematical Functions.

3. Sansone, Giovanni, Orthogonal Functions, Dover, reprinted in 2004

Spherical Harmonics

P=point on the sphere

$$P = (latitude, longitude) = (\lambda, \phi)$$

$$\lambda \in (0, 2\pi), \phi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$Y_{\ell s} = \begin{cases} \theta_{\ell s} \cos(s\lambda) P_{\ell s}(\sin\phi) & 0 \leq s \leq \ell \\ \theta_{\ell s} \sin(s\lambda) P_{\ell |s|}(\sin\phi) & -\ell \leq s < 0 \end{cases}$$

$$\ell = 0, 1, 2, \ldots, \quad P_{\ell s} = \text{ Legendre Polynomials}$$

The spherical harmonics are the eigenfunctions of the (horizontal) Laplacian Δ on the sphere:

$$\Delta f = \frac{1}{a^2}\left[\frac{1}{\cos^2\phi} f_{\lambda\lambda} + \frac{1}{\cos\phi}(\cos\phi f_\phi)_\phi\right]$$

$$\Delta Y_{\ell s} = -\ell(\ell + 1) Y_{\ell s}$$

and play the same role on the sphere as sines and cosines on the circle.

Spherical Harmonics (con't)

$$
\begin{aligned}
f &\in \mathcal{L}_2(Sphere) \\
f &\sim \sum_{\ell=0}^{\infty} \sum_{s=-\ell}^{\ell} f_{\ell s} Y_{\ell s}
\end{aligned}
$$

where

$$f_{\ell s} = \int_{Sphere} f(P) Y_{\ell s}(P) dP.$$

Positive Definite Functions on the Sphere

Letting P, P' be two points on the sphere. $B(P, P')$ given by

$$B(P,P') = \Sigma_{\ell s} \Sigma_{\ell' s'} b_{\ell s, \ell' s'} Y_{\ell s}(P) Y_{\ell' s'}(P')$$

will be positive definite if the matrix $\{b_{\ell s, \ell' s'}\}$ is positive definite. If this matrix is diagonal, and the entries only depend on ℓ, $b_{\ell s, \ell s} = \lambda_\ell$ then we have the famous addition formula for spherical harmonics:

$$\sum_{\ell s} \lambda_\ell Y_{\ell s}(P) Y_{\ell s}(P') = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \lambda_\ell P_\ell(\gamma(P,P'))$$

where P_ℓ is the ℓth Legendre polynomial, and $\gamma(P, P')$ is the cosine of the angle between P and P'.

The addition formula for spherical harmonics:

$$\sum_{s=-\ell}^{\ell} Y_{\ell s}(P) Y_{\ell s}(P') = \frac{2\ell+1}{4\pi} P_\ell(\gamma(P,P'))$$

is the generalization to the sphere of

$$sin(x)sin(x') + cos(x)cos(x') = cos(x - x')$$

Positive Definite Functions on the Sphere (continued)

According to a very old theorem of Schoenberg, an isotropic covariance on the sphere is always of the form

$$B(P, P') = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \lambda_\ell P_\ell(\gamma(P, P'))$$

with non-negative λ_ℓ. We describe two different families of isotropic covariances on the sphere that have been used in meteorological and biological applications. The first is based on a model of a stochastic process which is the solution of the mth iterated Laplacian driven by formal white noise, and the second is based on a diffusion model. They involve different rates of decay of the energy content of the processes. Probably one of the most important parameters in B is the implied rate of decay of energy with wavenumber. [Oversimplification of course].

First Family: Splines on the Sphere

Recall that $\Delta Y_{\ell s} = -\ell(\ell+1) Y_{\ell s}$.
Consider the zero-mean Gaussian stochastic process

$$X(P) = \sum_{\ell s} X_{\ell s} Y_{\ell s}(P)$$

with $\{X_{\ell s}\}$ independent, $\mathcal{N}(0, \frac{1}{[\ell(\ell+1)]^m})$. Then

$$\Delta^{m/2} X(P) = dW(P), \ \ [formal\ white\ noise].$$

Then

$$\begin{array}{rcl} EX(P)X(P') & = & \sum_{\ell s} \frac{1}{[\ell(\ell+1)]^m} Y_{\ell s}(P) Y_{\ell' s'}(P') \\ & \equiv & K_m(P, P'), \ \ say. \end{array}$$

Closed form expressions for a good approximation to $K_m, m = 3/2, 2, 5/2, \cdots, 6$ appear in Wahba 1981, 1982 `sphspl.pdf`.

$$f \sim \sum_{\ell=0}^{\infty} \sum_{s=-\ell}^{\ell} f_{\ell s} Y_{\ell s}$$

$$f_{\ell s} = \int_{Sphere} f(P) Y_{\ell s}(P) dP$$

$$\Delta f \sim \sum_{\ell=0}^{\infty} \sum_{s=-\ell}^{\ell} -\ell(\ell+1) f_{\ell s} Y_{\ell s}$$

$$J_m(f) = \int (\Delta^m f)^2 dP = \sum_{\ell=1}^{\infty} \sum_{s=-\ell}^{\ell} (\ell(\ell+1))^{2m} f_{\ell s}^2$$

$$K_m(P, P') = \sum_{\ell=1}^{\infty} \sum_{s=-\ell}^{\ell} \frac{1}{(\ell(\ell+1))^{2m}} Y_{\ell s}(P) Y_{\ell s}(P')$$

$$\lambda_{\ell s} = \frac{1}{[\ell(\ell+1)]^{2m}}$$

$$K_m(P, P') = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{[\ell(\ell+1)]^{2m}} P_\ell(\gamma(P, P'))$$

null space of J_m is $Y_{00}(P) \equiv 1$.
let $\mathcal{H}_m^0(S)$ be space of functions for which J_m is finite/constant functions.

In order to have a closed form expression for $K_m(P, P')$, it is necessary to sum the series

$$k_m(z) = \sum_{\nu=1}^{\infty} \frac{2\nu+1}{\nu^m(\nu+1)^m} P_\nu(z) \qquad (*)$$

To attempt to sum $(*)$ for m=2, we note that

$$\frac{2\nu+1}{\nu^2(\nu+1)^2} \equiv \frac{1}{\nu^2} - \frac{1}{(\nu+1)^2} \equiv \int_0^1 \log h(1-\frac{1}{h})h^\nu dh,$$
$$\nu = 1, 2, \cdots$$

Using the generating formula for Legendre polynomials (Sansone,p.169),

$$\sum_{\nu=1}^{\infty} h^\nu P_\nu(z) = (1-2hz+h^2)^{-\frac{1}{2}} - 1, \qquad (**)$$
$$-1 < h < 1$$

gives

$$k_2(z) = \int_0^1 \log h(1-\frac{1}{h})(\frac{1}{\sqrt{(1-2hz+h^2)}} - 1)dh$$

Thin plate pseudo-splines on the sphere

We seek a norm $Q_m^{\frac{1}{2}}(u)$ on $\mathcal{H}_m^0(S)$ which is topologically equivalent to $J_m^{\frac{1}{2}}(u)$ on $\mathcal{H}_m^0(S)$ and for which the reproducing kernel can be obtained in closed form convenient for computation.
Define

$$Q_m(u) = \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\xi_{\nu k}}, \quad u_{\nu k} = \int_S u(P) Y_{\nu k}(P) dP,$$

where

$$\xi_{\nu k} = [(\nu + \frac{1}{2})(\nu+1)(\nu+2)\cdots(\nu+2m-1)]^{-1}$$

Recall $\lambda_{\nu k} = \frac{1}{(\nu(\nu+1))^{2m}}$. Since

$$\frac{1}{m^{2m}\xi_{\nu k}} \le \frac{1}{\lambda_{\nu k}} \le \frac{1}{\xi_{\nu k}},$$

$$\nu = 1, 2, \cdots, \quad k = -\nu, \cdots, \nu, \quad m = 2, 3, \cdots,$$

we have

$$\frac{1}{m^{2m}} Q_m(u) \le J_m(u) \le Q_m(u), \quad u \in \mathcal{H}_m^0(S)$$

The norms $J_m^{\frac{1}{2}}(\cdot)$ and $Q_m^{\frac{1}{2}}(\cdot)$ are topologically equivalent on $\mathcal{H}_m^0(S)$. The reproducing kernel $R(P,P')$ for $\mathcal{H}_m^0(S)$ with norm $Q_m^{\frac{1}{2}}(\cdot)$ is then

$$
\begin{aligned}
&R(P,P') = R_m(P,P') \\
&= \sum_{\nu=1}^{\infty}\sum_{k=-\nu}^{\nu} \xi_{\nu k} Y_\nu^k(P) Y_\nu^k(P') \\
&= \frac{1}{2\pi}\sum_{\nu=1}^{\infty} \frac{1}{(\nu+1)\cdots(\nu+2m-1)} P_\nu(\gamma(P,P')).
\end{aligned}
$$

A closed form expression can be obtained for $R_m(P,P')$ as follows. Use the fact that

$$
\frac{1}{r!}\int_0^1 (1-h)^r h^\nu dh \equiv \frac{1}{(\nu+1)\cdots(\nu+r+1)}, \quad r = 0,1,2,\cdots.
$$

Then by using the generating function $(**)$ for the Legendre polynomials we have

$$R_m(P,P') = \frac{1}{2\pi}\sum_{\nu=1}^{\infty}\frac{1}{(\nu+1)\cdots(\nu+2m-1)}P_\nu(z)$$
$$= \frac{1}{2\pi}[\frac{1}{(2m-2)!}q_{2m-2}(z) - \frac{1}{(2m-1)!}].$$

where

$$z = \gamma(P,P')$$

(cosine of the angle between P and P'), and

$$q_m(z) = \int_0^1 (1-h)^m(1-2hz+h^2)^{-\frac{1}{2}}dh,$$
$$m = 0,1,\cdots.$$

Formulas for $\int h^m(1-2hz+h^2)^{-\frac{1}{2}}dh, m = 0,1,2$ and recursion formulas for general m in terms of the formulars for $m=1$ and $m=2$ can be found in B. O. Pierce and R. M. Foster, A Short Table of Integrals, Grun and Co, 1956[pp. 165,174,177,196]. Macsyma evaluated $q_m(z)$ recursively. The results appear in the following table.

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ERRATUM: SPLINE INTERPOLATION AND SMOOTHING ON THE SPHERE*

GRACE WAHBA†

Table 1 contains several misprints in lines q [6], q [7] and q [8]. The correct table appears below.

TABLE 1

$$q_m(z)=\int_0^1 (1-h)^m(1-2hz+h^2)^{-1/2}\,dh, \qquad m=0,1,\cdots,10.$$

Key. $q[m]=q_m(z)$, $A=\ln(1+1/\sqrt{W})$, $C=2\sqrt{W}$, $W=(1-z)/2$

q[0];

$$A$$

q[1];

$$2AW - C + 1$$

q[2];

$$\frac{A(12W^2 - 4W) - 6CW + 6W + 1}{2}$$

q[3];

$$\frac{A(60W^3 - 36W^2) + 30W^2 + C(8W - 30W^2) - 3W + 1}{3}$$

q[4];

$$\frac{A(840W^4 - 720W^3 + 72W^2) + 420W^3 + C(220W^2 - 420W^3) - 150W^2 - 4W + 3}{12}$$

q[5];

$$(A(7560W^5 - 8400W^4 + 1800W^3) + 3780W^4$$
$$+ C(-3780W^4 + 2940W^3 - 256W^2) - 2310W^3 + 60W^2 - 5W + 6)/30$$

q[6];

$$(A(27720W^6 - 37800W^5 + 12600W^4 - 600W^3) + 13860W^5$$
$$+ C(-13860W^5 + 14280W^4 - 2772W^3) - 11970W^4 + 1470W^3 + 15W^2 - 3W + 5)$$
$$/30$$

* This Journal, 2 (1981), pp. 5–16.

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continued

Thus, for example,$R(P, P')$ for m=2 involves q_2 and , from the table,

$$q[2] = \frac{A(12W^2 - 4W) - 6CW + 6W + 1}{2},$$

(where A,C and W are defined on the Table) giving

$$q_2(z) = \frac{1}{2}\{ln(1 + \sqrt{\frac{2}{1-z}})[12(\frac{1-z}{2})^2 - 4\frac{(1-z)}{2}]$$
$$- 12(\frac{1-z}{2})^{3/2} + 6(\frac{1-z}{2}) + 1\}$$

Note that $q[0]$ which appears in the $m = 1$ case does not lead to a proper rk since $q_0(1)$ is not finite. However, a proper rk exists for any $m > 1$, and the table can be used to define q_{2m-2} for $m = \frac{3}{2}, 2, \frac{5}{2}, \cdots, 6$.

Splines on the Sphere, continued.

Figure 1: Sample Fourier coefficients.
Figure 2: $\lambda_\ell = \frac{1}{[\sum_{j=0}^{2} a_j [\ell(\ell+1)]^j]^2}$.

$$\sum_{j=0}^{2} \alpha_j \Delta^j X(P) = dW(P).$$

Figure 3: Correlation function corresponding to the covariance for Figure 2.
Figure 4: A sample correlation function from another data set.

Alternatively, if $\lambda_{\ell s} = \sum_m \frac{b_m}{\ell(\ell+1)^{2j}}$, then the corresponding covariance is $\sum_m b_m R_m(P, P')$, and the closed form approximating expressions in Wahba(1982) could be used.

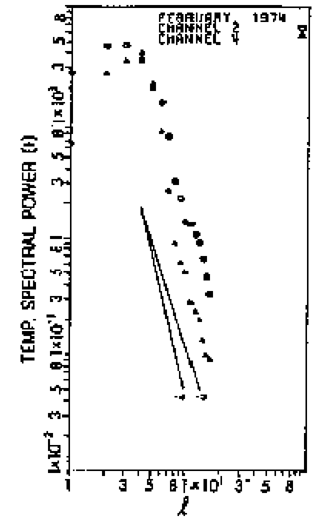


Figure 1: Temperature Spectral Power (ℓ).

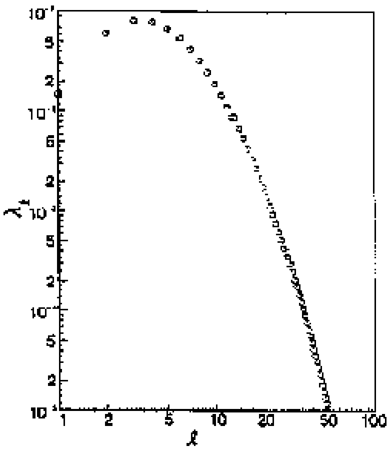


Figure 2: Idealized λ_ℓ.

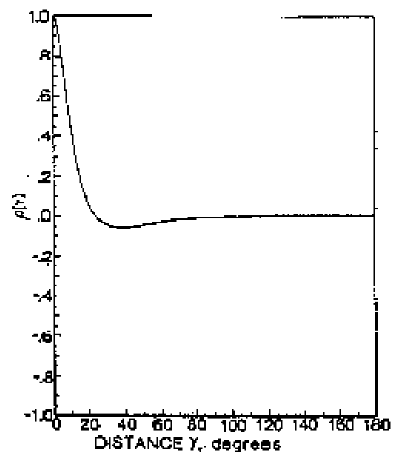


Figure 3: Correlation function for the (λ_ℓ) of Fig. 2.

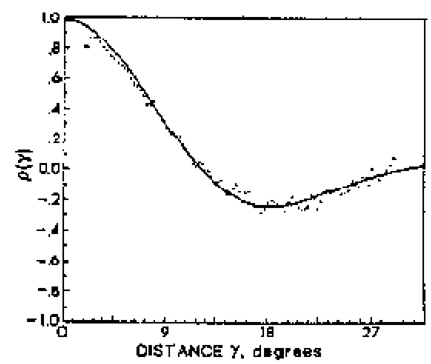


Figure 4: Sample Correlation Function.

Fig1. Temperature Spectral Power, as a function of ℓ.
Fig2. Idealized Power Spectrum, as a function of ℓ.
Fig3. Correlation for the power spectrum of Fig2. as a function of distance, in degrees.
Fig4. Sample correlation vs distance, from Fig1. data.

Second Family: Diffusion Models on the Sphere.

Consider the diffusion equation

$$\frac{\partial f}{\partial t} - \kappa \Delta f(P) = 0$$

Letting $f(P,t) = \sum_{\ell s} f_{\ell s}(t) Y_{\ell s}(P)$, f will satisfy the diffusion equation if

$$\frac{df_{\ell s}}{dt} = -\kappa \ell(\ell+1) f_{\ell s}(t),$$

so that $f(P,0)$ ”diffuses” in time T to

$$f(P,T) = \sum_{\ell s} f_{\ell s}(0) e^{-\kappa \ell(\ell+1)T} Y_{\ell s}(P).$$

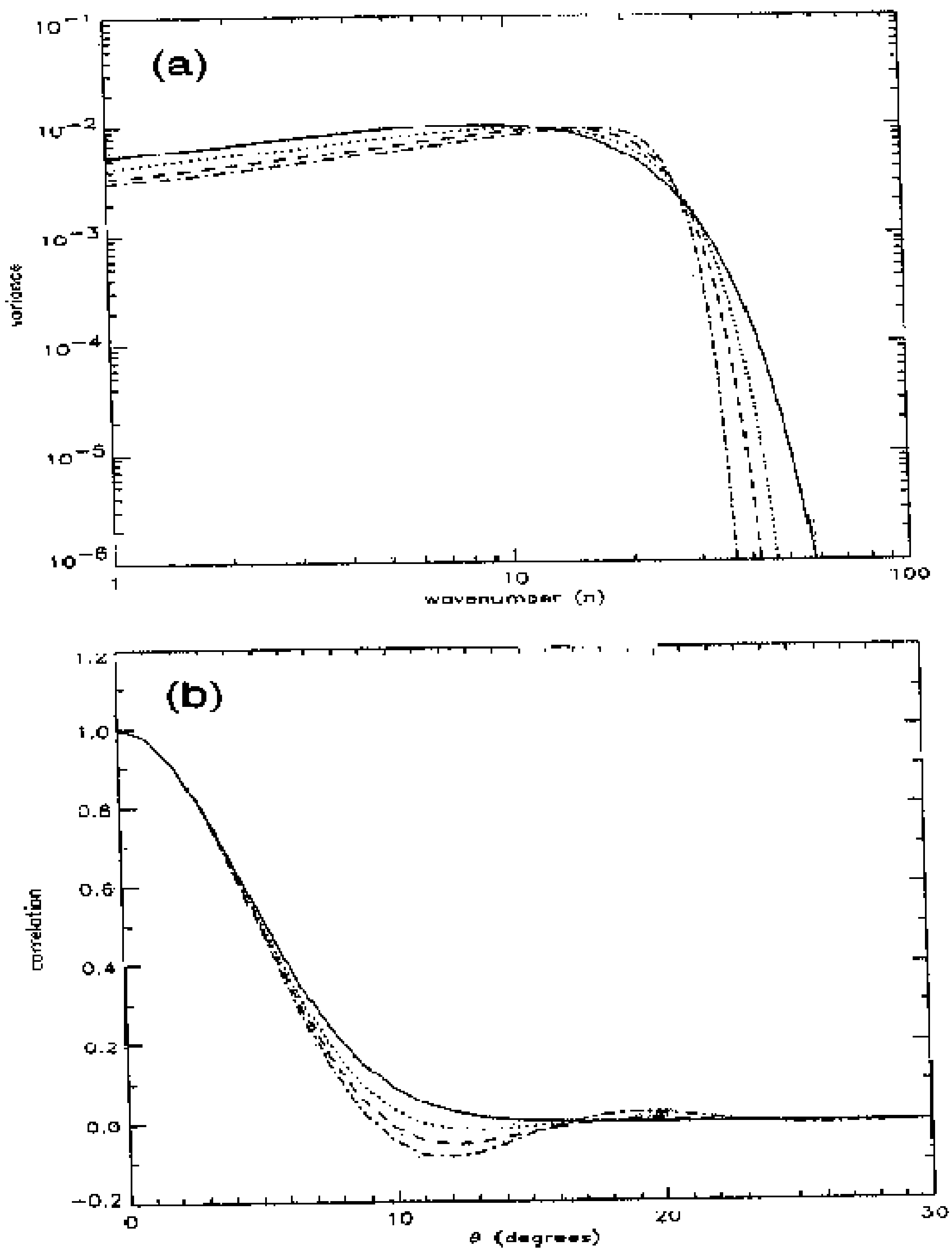
Courtier and Weaver, QJRM(2001) used this argument to propose the isotropic covariance model with $\lambda_{\ell s} = e^{-\kappa \ell(\ell+1)}$, or, more generally they proposed considering the more general p.d.e.

$$\frac{\partial f}{\partial t} + \sum_j \kappa_j (-\Delta)^j f(P) = 0,$$

which leads to $\lambda_\ell = e^{-\sum_j \kappa_j [\ell(\ell+1)]^j}$.

Diffusion Models on the Sphere (continued).

Figure (a): (Weaver and Courtier) gives several different plots of λ_ℓ. The heavy line corresponds to a $j = 1$ model and the dotted lines correspond to particular $j = 2$ and $j = 3$ models, all scaled to have the same length scale.
Figure (b): Coresponding correlation functions.


(a)
variance
wavenumber (n)
(b)
correlation
θ (degrees)

(Optional, remark) The sphere is a compact Riemannian manifold, (Riemannian manifold: has an inner product on the tangent space at each point that varies smoothly.. on the sphere the tangent space is 2-dimensional Euclidean space). kim.2000.pdf discusses a definition of splines on a Riemannian manifold. Note the role of the Laplacian and its eigenfunctions).