# Splines on Riemannian Manifolds and a Proof of a Conjecture by Wahba

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**Abstract**

This paper extends spline methods to compact Riemannian manifolds in an *rkhs* setting. The approach is to use the mathematical framework of *rkhs*, along with integrating spectral geometry associated with compact Riemannian manifolds. This combination affirmatively answers a conjecture made by Wahba (1981) that spline interpolation and smoothing available for the 2–sphere can be generalized to compact Riemannian manifolds. Applications to higher dimensional spheres and rotation matrices are also exhibited.

**Proposed Running Title:** Splines on Manifolds and Wahba

**Key Words**: Addition theorem; Eigenfunctions; Eigenvalues; Laplace-Beltrami operator; Regularization; Reproducing kernel Hilbert space; Sobolev space; Zeta function.

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# 1 Introduction

The subject of splines is an active area of mathematical research and there are many references on the subject as well as many different points of view. In a functional analysis framework, a very powerful approach is to investigate splines through the theory of reproducing kernel Hilbert spaces (*rkhs*). A good survey, along with applications is summarized succinctly in Wahba (1990); see also the references contained therein.

In terms of developing splines from the *rkhs* approach, Wahba (1990) confines her domain to a standard class, which, although is beneficial for most practical applications, nevertheless, is restrictive. Indeed, in terms of the mathematical foundation that the *rkhs* offers, it is possible that extensions of spline methods to compact Riemannian manifolds can be made if one adheres to *rkhs* theory and use aspects of spectral geometry. In fact, Wahba (1981) successfully implements such a strategy for the 2–sphere and conjectures that the methods thus developed "can no doubt be generalized to establish splines associated with the Laplace-Beltrami operator on compact Riemannian manifolds", Wahba (1981, 9). We will understand the latter statement as *Wahba's conjecture*. A formal statement will later be provided.

It turns out that the 2–sphere is an excellent prototypical example that can provide insights into compact Riemannian manifolds in general. The pursuit of this paper therefore, is to show that one can do such an analysis in the more general setting using methods from spectral geometry. In particular, this paper affirmatively answers Wahba's conjecture as well as display the rich mathematical foundation that the *rkhs* framework provides in terms of practical spline methodology over more general domains.

Since Wahba's paper on spline interpolation and smoothing on the 2–sphere, a generalization to the hypersphere is carried out in Taijeron, Gibson and Chandler (1994). They consider more general operators of which the Laplace-Beltrami operator is a special case. Again, using methods from spectral geometry, one can accomodate the results of Taijeron, Gibson and Chandler (1994) in the more general setting, consequently, we will adopt the latter's approach as the outline to the generalization to compact Riemannian manifolds.

We now provide a summary of what is to follow.

In Section 2, we briefly review some notation and geometric preliminaries. Putting this into the *rkhs* framework then follows which includes a discussion of reproducing kernels. Some specific results for reproducing kernels can be made through a beautiful generalization of the addition formula for spherical harmonics by Ginè (1975b). In particular, one can express the reproducing kernel in terms of zonal functions when the manifold is in addition homogeneous. Even more can be said if the manifold in question is a Lie group. In this case we would use the representation theory for compact Lie groups.

Once this is complete, the construction of the *rkhs* in a geometric setting basically answers Wahba's conjecture. In particular, the characterizing minimization problems of interpolation and smoothing has a spline solution with respect to a general class of operators which includes the Laplace-Beltrami operator. This is discussed in Section 3 along with affirmatively answering Wahba's conjecture with respect to the Laplace-Beltrami operator. We also outline how one of the key conditions in providing a spline solution is intricately connected to some classical results on the zeta function of the Laplace-Beltrami operator. Due to certain technical conditions, we will present the proofs to the results in Section 3, separately in Section 6. Here we go over the regularization procedure of Taijeron, Gibson and Chandler (1994) for the hypersphere and extend it to compact Riemannian manifolds. At this point we explicitly use the Riemannian structure through the Riemannian metric to intrinsically regularize the problem.

The basic examples of compact manifolds are the higher dimensional hyperspheres $S^{m-1}$ and the group of $N \times N$ rotation matrices $SO(N)$. Each of course have there explicit constructions and this is presented as examples in Section 4. A brief discussion is provided in Section 5, highlighting the achievements of the paper as well as some comments on implementation.

## 2 Compact Manifolds Preliminaries

We will first present the notation and some preliminaries. Aspects of geometry and Lie groups that will be used come from standard sources, for example, Spivak (1970), Helgason (1978, 1984), Bröcker and tom Dieck (1985) and Fegan(1991).

Let $\mathcal{M}$ be an $m-$dimensional compact Riemannian manifold. Consider the Riemannian structure $\{g_p(\cdot,\cdot) : p \in \mathcal{M}\}$ and let $dx$ be the normalized volume element of $\mathcal{M}$ associated with this structure. We will in addition assume that the manifold is connected and without boundary, $\partial\mathcal{M} = \emptyset$, although one could generalize the following arguments to certain boundary conditions, for example, von Neumann boundary conditions.

Consider $\gamma(t)$ a smooth curve in $\mathcal{M}$, with $t \in [a,b]$ and let $\gamma'(t)$ denote it's first derivative. Then the length of $\gamma$ is defined through the Riemannian structure as

$$l(\gamma) = \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2} dt.$$

Since we are assuming that $\mathcal{M}$ is connected, hence for any two points $p, q \in \mathcal{M}$ we can find a curve in $\mathcal{M}$ that joins them in $\mathcal{M}$, we can define a metric on $\mathcal{M}$ by

$$\rho(p,q) = \inf\{l(\gamma) : \gamma \text{ joining } p \text{ and } q\}, \tag{2.1}$$

$p, q \in \mathcal{M}$. This metric is called the Riemannian metric which makes $(\mathcal{M}, \rho)$, a metric space. This of course is the intuitive and classical definition. The more modern approach is to view the Riemannian metric as an inner product on the tangent bundle of $\mathcal{M}$. Spivak (1970), provides a lively discussion on this matter in chapter 9.

Let $C^\infty(\mathcal{M})$ be the space of real valued infinitely differentiable continuous functions on $\mathcal{M}$. Denote by $\Delta$, the Laplace-Beltrami operator on $\mathcal{M}$. It is understood that $\Delta$ is an elliptic self-adjoint second order differential operator on $C^\infty(\mathcal{M})$, hence by the spectral theorem for compact operators, the eigenfunctions of $\Delta$ is a complete orthonormal basis for $L^2(\mathcal{M})$.

Let $\phi_\lambda$ and $\lambda$ be an eigenvector and the corresponding eigenvalue of $\Delta$, respectively. For $\mathbb{N} = \{0, 1, 2, \ldots\}$, note that there are countably many $\lambda_k \geq 0$, $k \in \mathbb{N}$ with no upper bound. This means that for each $\lambda_k$, we can denote a corresponding eigenfunction (which in general will occur with multiplicity) by $\phi_{\lambda_k} = \phi_k$, $k \in \mathbb{N}$. Furthermore, we will use the convention that $\lambda_0 = 0$ with $\phi_0 = 1$ and that $\lambda_k \leq \lambda_{k+1}$ for $k \in \mathbb{N}$.

For functions $f : \mathcal{M} \to \mathbb{R}$, let $L^2(\mathcal{M})$ denote the space of square integrable functions. Let $\mathcal{E}_k \subset L^2(\mathcal{M})$, $k \in \mathbb{N}$, denote the eigenspace associated with the eigenvalue $\lambda_k$, $k \in \mathbb{N}$. The dimension of $\mathcal{E}_k$ will be denoted by $\dim \mathcal{E}_k$ for $k \in \mathbb{N}$. The multipicity of eigenvectors whose eigenvalues are less than a certain constant is determined by Weyl's formula

$$\lim_{T\to\infty} T^{-m/2} \#\{\lambda_k < T\} = \frac{\mathrm{vol}(\mathcal{M})}{(2\sqrt{\pi})^m \Gamma(1 + m/2)}, \tag{2.2}$$

where $\mathrm{vol}(\mathcal{M})$ denotes the $m-$dimensional volume of $\mathcal{M}$ and $\Gamma(\cdot)$ is the gamma function, see Minakshisundaram and Pleijel (1949). We note that if $\phi_k$ is an eigenfunction of $\Delta$, then so is $\overline{\phi}_k$ where overbar denotes complex conjugation. Consequently, a real basis for $L^2(\mathcal{M})$ can be chosen.

For $h \in L^2(\mathcal{M})$, the eigenfunction expansion will be defined by

$$h = \sum_{k=0}^{\infty} \sum_{\mathcal{E}_k} \hat{h}_k \phi_k, \quad \text{where} \quad \hat{h}_k = \int_{\mathcal{M}} h \overline{\phi}_k, \tag{2.3}$$

for $k \in \mathbb{N}$. We note that summation over $\mathcal{E}_k$ means over all eigenfunctions $\phi_k$ in the eigenspace $\mathcal{E}_k$, $k \in \mathbb{N}$.

## 2.1 Reproducing kernel Hilbert space

Let $\{a_k : k \in \mathbb{N}\}$ be a sequence of numbers. For $h \in L^2(\mathcal{M})$, define the norm

$$\|h\|^2_{\mathcal{A}} = \sum_{k=0}^{\infty} \sum_{\mathcal{E}_k} |a_k|^2 |\hat{h}_k|^2. \tag{2.4}$$

Let $H_{\mathcal{A}}(\mathcal{M})$ be the vector space completion of $C^\infty(\mathcal{M})$ with respect to the norm (2.4). This allows us to define the operator $\mathcal{A} : H_{\mathcal{A}}(\mathcal{M}) \to L^2(\mathcal{M})$

$$\mathcal{A}h = \sum_{k=0}^{\infty} \sum_{\mathcal{E}_k} a_k \hat{h}_k \phi_k. \tag{2.5}$$

We note that if $\mathcal{A} = \Delta^{s/2}$, then denote by $H_{\Delta^{s/2}}(\mathcal{M}) = H_s(\mathcal{M})$, the Sobolev space of order $s$. Furthermore, $\Delta^{s/2} u = \sum \lambda_k^{s/2} \hat{h}_k \phi_k$ and

$$\int_{\mathcal{M}} |\Delta^{s/2} u|^2 = \sum_k \sum_{\mathcal{E}_k} \lambda_k^s |\hat{h}_k|^2, \tag{2.6}$$

see Lemma 4.1 Hendriks (1990).

Let $\mathcal{P}_k : H_{\mathcal{A}}(\mathcal{M}) \to \mathcal{E}_k$, $k \in \mathbb{N}$ be the projection operator onto the $k-$th eigenspace and denote by $\mathbb{N}_0^{\mathcal{A}} = \{k \in \mathbb{N} : \mathcal{A}(\mathcal{P}_k) = 0\}$. Then writing

$$H^0_{\mathcal{A}}(\mathcal{M}) = \bigoplus_{k \in \mathbb{N}_0^{\mathcal{A}}} \mathcal{E}_k, \quad H^1_{\mathcal{A}}(\mathcal{M}) = \bigoplus_{k \notin \mathbb{N}_0^{\mathcal{A}}} \mathcal{E}_k, \tag{2.7}$$

we have the decomposition

$$H_{\mathcal{A}}(\mathcal{M}) = H^0_{\mathcal{A}}(\mathcal{M}) \oplus H^1_{\mathcal{A}}(\mathcal{M}). \tag{2.8}$$

Let $u = u_0 + u_1$ and $v = v_0 + v_1$ with $u_0, v_0 \in H^0_{\mathcal{A}}(\mathcal{M})$ and $u_1, v_1 \in H^1_{\mathcal{A}}(\mathcal{M})$. Then, let

$$\langle u_0, v_0 \rangle^0_{\mathcal{A}} = \sum_{k \in \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} (\hat{u_0})_k (\overline{\hat{v_0}})_k, \quad \langle u_1, v_1 \rangle^1_{\mathcal{A}} = \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^2 (\hat{u_1})_k (\overline{\hat{v_1}})_k$$

$$\text{and} \quad \langle u, v \rangle_{\mathcal{A}} = \langle u_0, v_0 \rangle^0_{\mathcal{A}} + \langle u_1, v_1 \rangle^1_{\mathcal{A}},$$

be the inner products of $H^0_{\mathcal{A}}(\mathcal{M})$, $H^1_{\mathcal{A}}(\mathcal{M})$ and $H_{\mathcal{A}}(\mathcal{M})$, respectively.

For $p, q \in \mathcal{M}$, define

$$K_0(p,q) = \sum_{k \in \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} \phi_k(p) \overline{\phi}_k(q), \tag{2.9}$$

$$K_1(p,q) = \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2} \phi_k(p) \overline{\phi}_k(q), \tag{2.10}$$

and

$$K(p,q) = K_0(p,q) + K_1(p,q). \tag{2.11}$$

**Lemma 2.1** *Let $\mathcal{M}$ be a compact connected Riemannian manifold. Suppose $\mathbb{N}_0^{\mathcal{A}}$ is finite and*

$$\sup_{p\in\mathcal{M}} \sum_{k\notin\mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2}|\phi_k(p)|^2 < \infty,$$

*for $k = 0, 1, \ldots$. Then, $H^0_{\mathcal{A}}(\mathcal{M})$, $H^1_{\mathcal{A}}(\mathcal{M})$ and $H_{\mathcal{A}}(\mathcal{M})$ are* rkhs *with reproducing kernels $K_0$, $K_1$ and $K$, respectively.*

**Proof.** For each $p \in \mathcal{M}$, $K_0(p, \cdot) \in H^0_{\mathcal{A}}(\mathcal{M})$. Furthermore, for $u_0 \in H^0_{\mathcal{A}}(\mathcal{M})$, we have

$$\langle K_0(p, \cdot), u_0\rangle^0_{\mathcal{A}} = \sum_{k\in\mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} \phi_k(p)(\hat{u}_0)_k = u_0(p).$$

Hence, $K_0$ is a reproducing kernel and $H^0_{\mathcal{A}}(\mathcal{M})$ is an *rkhs*. For $K_1(p, \cdot)$ we note that

$$\left(\|K_1\|^1_{\mathcal{A}}\right)^2 = \sum_{k\notin\mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2}|\phi_k(p)|^2 \le \sup_{p\in\mathcal{M}} \sum_{k\notin\mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2}|\phi_k(p)|^2 < \infty,$$

by assumption. Thus $K_1(p, \cdot) \in H^1_{\mathcal{A}}(\mathcal{M})$ and

$$\langle K_1(p, \cdot), u_1\rangle^1_{\mathcal{A}} = \sum_{k\notin\mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} \phi_k(p)(\hat{u}_1)_k = u_1(p),$$

for all $u_1 \in H^1_{\mathcal{A}}(\mathcal{M})$. Consequently, $K_1$ is a reproducing kernel and $H^1_{\mathcal{A}}(\mathcal{M})$ is an *rkhs*. Thus it follows that $K$ is a reproducing kernel and that $H_{\mathcal{A}}(\mathcal{M})$ is an *rkhs*. □

## 2.2 Reproducing kernel on homogeneous spaces

In a practical setting, $\mathcal{M}$ is usually equipped with certain symmetries. In light of this as well as a very beautiful addition formula available on manifolds with certain symmetries, we will at times impose an additional technical condition on $\mathcal{M}$. A Riemannian manifold is homogeneous if its group of isometries $\mathcal{G}$, acts transitively on $\mathcal{M}$, where by the latter, we mean that for every $p, q \in \mathcal{M}$, there exits a $g \in \mathcal{G}$ such that $p = g \cdot q$. For every $p_0 \in \mathcal{M}$, let $\mathcal{G}_{p_0} = \{g \in \mathcal{G} : g \cdot p_0 = p_0\}$ denote the isotropy subgroup of $p_0$. It is well known that if $\mathcal{M}$ is a homogeneous compact connected Riemannian manifold, then for every $p \in \mathcal{M}$, $\mathcal{G}_p$ is a closed subgroup of $\mathcal{G}$ and there exists a diffeomorphism of $\mathcal{G}/\mathcal{G}_p \simeq \mathcal{M}$. The classical example is the diffeomorphism of the 2–sphere $S^2$ with the quotient set of $3 \times 3$ rotation matrices modulo $2 \times 2$ rotation matrices $SO(3)/SO(2)$. A differentiable function $f : \mathcal{M} \to \mathbb{R}$ is called a zonal function with respect to $p_0 \in \mathcal{M}$ if it is constant on the isotropy subgroup $\mathcal{G}_{p_0}$.

What is of practical importance is the evaluation of (2.10). For a homogeneous space, a beautiful generalization of the addition formula for spherical harmonics is available, see Giné (1975a, 1975b). The consequence is that the evaluation of (2.10) can be made in terms of zonal functions on $\mathcal{M}$. We have the following.

**Lemma 2.2** *Let $\mathcal{M}$ be a compact connected homogeneous Riemannian manifold. For $x_0 \in \mathcal{M}$ fixed, $\Psi : \mathcal{M} \to \mathcal{M}$ an isometry and $f_{x_0}^{\lambda_k} : \mathcal{E}_k \to \mathbb{R}$ a zonal function with respect to $x_0 \in \mathcal{M}$,*

$$K_1(p,q) = \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} |a_k|^{-2} (\mathrm{dim}\mathcal{E}_k)^{1/2} f_{x_0}^{\lambda_k}(\Psi_{p,x_0}(q)),$$

*where $p, q \in \mathcal{M}$.*

**Proof.** Fix an eigenspace $\mathcal{E}_k$, say. Then in $\mathcal{E}_k$, we have by Theorem 3.2 of Giné (1975b),

$$\sum_{\mathcal{E}_k} \phi_k(p)\overline{\phi}_k(q) = (\dim \mathcal{E}_k)^{1/2} f_{x_0}^{\lambda_0}(\Psi_{p,x_0}(q)),$$

where $x_0 \in \mathcal{M}$ is fixed, $\Psi : \mathcal{M} \to \mathcal{M}$ is an isometry and $f_{x_0}^{\lambda_k} : \mathcal{E}_k \to \mathbb{R}$ is a zonal function with respect to $x_0 \in \mathcal{M}$. Therefore,

$$\begin{aligned} K_1(p,q) &= \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2} \phi_k(p)\overline{\phi}_k(q) \\ &= \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} |a_k|^{-2} \sum_{\mathcal{E}_k} \phi_k(p)\overline{\phi}_k(q) \\ &= \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} |a_k|^{-2} (\mathrm{dim}\mathcal{E}_k)^{1/2} f_{x_0}^{\lambda_k}(\Psi_{p,x_0}(q)). \quad \square \end{aligned}$$

Furthermore, we note that for $\mathcal{M}$ homogeneous,

$$\sum_{\mathcal{E}_k} |\phi_k(p)|^2 = \dim \mathcal{E}_k, \quad \text{for all} \quad p \in \mathcal{M}, \tag{2.12}$$

for $k = 0, 1, \ldots$, see Theorem 3.2 Giné (1975b). Consequently, the condition

$$\sup_{p \in \mathcal{M}} \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} \sum_{\mathcal{E}_k} |a_k|^{-2} |\phi_k(p)|^2 < \infty, \tag{2.13}$$

for $k = 0, 1, \ldots$ in Lemma 2.1 can be replaced by

$$\sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} |a_k|^{-2} \dim \mathcal{E}_k < \infty, \tag{2.14}$$

for $k = 0, 1, \ldots$.

An even finer structure is to assume that for $x_1, y_1, x_2, y_2 \in \mathcal{M}$, with $\rho(x_1, y_1) = \rho(x_2, y_2)$, there exists a $g \in \mathcal{G}$ such that $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$. In such a case, $\mathcal{M}$ is called two-point homogeneous. Then $f_{x_0}^{\lambda_k}(\Psi_{p,x_0}(q))$ can be replaced by a uniquely determined real valued function $h^{\lambda_k}(\rho(p,q))$ where $h^{\lambda_k}(r) = f_{x_0}^{\lambda_k}(\Psi_{p,x_0}(x))$ for every $x \in \mathcal{M}$ with $\rho(x, x_0) = r$. Thus, for a two-point homogeneous space,

$$K_1(p,q) = \sum_{k \notin \mathbb{N}_0^{\mathcal{A}}} |a_k|^{-2} (\mathrm{dim}\mathcal{E}_k)^{1/2} h^{\lambda_k}(\rho(p,q)),$$

where $p, q \in \mathcal{M}$. Notice that in this case the reproducing kernel depends only on the Riemannian distance between a pair of points.

## 2.3 Reproducing kernel on Lie groups

If in addition the manifold has a group structure with the group action and inverse mapping being continuous hence is a Lie group, additional refinements on (2.10) can be made.

Let $\mathcal{G}$ be a compact connected Lie group and fix once and for all, a maximal torus $\mathbb{T}$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the corresponding Lie algebras and denote by $\mathfrak{t}^*$ the dual space of $\mathfrak{t}$ possessing the Weyl group invariant inner product. Let $J \subset \mathfrak{t}^*$ be the fundamental Weyl chamber and denote by $\Phi$, the set of real roots. Let $\Phi_+ = \{\alpha \in \Phi : \langle \alpha, \beta \rangle > 0, \beta \in J\}$ be the set of positive roots. Finally denote by $I^* \subset \mathfrak{t}^*$ the integral portion of $\mathfrak{t}^*$. Thus we can define $\bar{J} \cap I^*$ which will be used below for indexing. We are using $\bar{J}$ to denote set theoretic closure, however this conflicts with the notation for complex conjugation. This is unfortunate however the set theoretic closure will only be used on $J$ and the latter will never be used to denote a complex quantity.

Consider an irreducible representation $U : \mathcal{G} \to \mathrm{Aut}(V)$, where $V$ is some finite dimensional vector space and $\mathrm{Aut}(V)$ are the automorphisms of $V$. Then the collection of inequivalent irreducible representations of $\mathcal{G}$ can be enumerated as $\{U_\nu : \nu \in \bar{J} \cap I^*\}$, see Bröcker and tom Dieck (1985, 242). The irreducible characters are defined by $\chi_\nu = \mathrm{tr}\, U_\nu$ and the dimension of the irreducible representations are $d(\nu+\rho) = \prod_{\alpha \in \Phi_+} \langle \alpha, \nu+\rho \rangle / \langle \alpha, \rho \rangle$ for $\nu \in \bar{J} \cap I^*$ where $\rho = 2^{-1} \sum_{\alpha \in \Phi_+} \alpha$, the half sum of the positive roots and the inner product is induced by the Killing form.

The Killing form also induces a Riemannian structure on $\mathcal{G}$. Consequently, let $\Delta$ be the Laplace-Beltrami operator on $\mathcal{G}$. The components of the irreducible representations are the eigenfunctions of $\Delta$ so that

$$\left\{ \sqrt{d(\nu+\rho)}\, U_\nu : \nu \in \bar{J} \cap I^* \right\} \tag{2.15}$$

is a complete orthonormal basis of $L^2(\mathcal{G})$. We note that the eigenvalues are

$$\lambda_\nu = \|\nu + \rho\|^2 - \|\rho\|^2 \tag{2.16}$$

for $\nu \in \bar{J} \cap I^*$ where the norm is with respect to the Killing form. The multiplicity of the eigenfunctions with respect to a fixed eigenvalue is therefore $d^2(\nu+\rho)$ for $\nu \in \bar{J} \cap I^*$. Weyl's formula (2.2) for $\mathcal{G}$ is thus

$$\lim_{T\to\infty} T^{-\dim \mathcal{G}/2} \sum_{\lambda_\nu < T} d^2(\nu+\rho) = \frac{\mathrm{vol}\,(\mathcal{G})}{(2\sqrt{\pi})^{\dim \mathcal{G}} \Gamma(1 + \dim \mathcal{G}/2)}, \tag{2.17}$$

see Minakshisundaram and Pleijel (1949), Giné (1975), or Hendriks (1990).

The general formula for (2.10) for $\mathcal{G}$ is given in the following.

**Lemma 2.3** *Suppose $\mathcal{G}$ is a connected compact Lie group. Then*

$$K_1(p,q) = \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho)\chi_\nu(pq^{-1}),$$

*where $p,q \in \mathcal{G}$ and $(\bar{J} \cap I^*)^0_{\mathcal{A}}$ is in bijective correspondence with $\mathbb{N}_0^{\mathcal{A}}$.*

**Proof.** We note that

$$\begin{aligned} K_1(p,q) &= \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho) \operatorname{tr} U_\nu(p) \overline{U}^t_\nu(q) \\ &= \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho) \operatorname{tr} U_\nu(p) U_\nu(q^{-1}) \\ &= \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho) \operatorname{tr} U_\nu(pq^{-1}) \\ &= \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho) \chi_\nu(pq^{-1}), \end{aligned} \tag{2.18}$$

where superscript $t$ denotes transposition, $\overline{U}^t_\nu(q) = U_\nu(q^{-1})$, and one uses the fact that $U_\nu$ is a group homomorphism. □

We note that Lemma 2.3 will be true for all connected compact Lie groups. Thus the only issues are the differences in the root structure of $\mathcal{G}$ which are classified by the Dynkin diagrams, see Bröcker and tom Dieck (1985, 212). In addition, the irreducible characters can be calculated through the Weyl character formula, see Bröcker and tom Dieck (1985, 244).

We can further refine Lemma 2.3 to

$$K_1(p,q) = \sum_{\nu \notin (\bar{J} \cap I^*)^0_{\mathcal{A}}} |a_\nu|^{-2} d(\nu+\rho) \sum_{\chi_\nu(e) = d(\nu+\rho)} \chi_\nu(pq^{-1}), \tag{2.19}$$

for $p,q \in \mathcal{G}$. Furthermore, because of the invariance of (2.19) with respect to the group action $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and because the irreducible characters of any compact Lie group $\mathcal{G}$ is defined on $\mathbb{T}$, the number of variables that $K(p,q)$ would depend on would be $\dim \mathbb{T}$. In addition, if $\mathcal{A} = \Delta^{s/2}$, then by (2.16), we would set $|a_\nu|^{-2} = \left(\|\nu+\rho\|^2 - \|\rho\|^2\right)^{-s}$ in (2.19).

# 3 Splines on Manifolds

We are now ready to state the main results of this paper. All proofs will be deferred to Section 6.

**Definition 3.1** *The* interpolation problem *is to find $u \in H_{\mathcal{A}}(\mathcal{M})$ that minimizes*

$$\left(\|u\|_{\mathcal{A}}^{1}\right)^2 = \int_{\mathcal{M}} |\mathcal{A}u|^2$$

*subject to the constraint*

$$u(p_i) = z_i, \quad i = 1, \dots, n,$$

*where $\mathcal{A}$ is the operator defined by (2.5), $p_i \in \mathcal{M}$ and $z_i \in \mathbb{R}$ for $i = 1, \dots, n$.*

**Definition 3.2** *The* smoothing problem *is*

$$\min_{u \in H_s(\mathcal{M})} \frac{1}{n} \sum_{i=1}^{n} \left(u(p_i) - z_i\right)^2 + \xi \left(\|u\|_{\mathcal{A}}^{1}\right)^2,$$

*where $\xi > 0$, $p_i \in \mathcal{M}$ and $z_i \in \mathbb{R}$ for $i = 1, \dots, n$.*

We will need the following notation. Let $p_1, \dots, p_n, q \in \mathcal{M}$. Define the $n \times n$ matrix

$$K_{n,\xi} = [K(p_i, p_j)] + n\xi I, \tag{3.1}$$

where $i, j = 1, \dots, n$, $\xi \geq 0$ and $I$ is the $n \times n$ identity matrix. Let

$$m_0 = \sum_{k \in \mathbb{N}_0^{\mathcal{A}}} \dim \mathcal{E}_k. \tag{3.2}$$

Enumerate $\mathbb{N}_0^{\mathcal{A}} = \{k_1, \dots, k_r\}$ and define the $n \times m_0$ matrix

$$S_{m_0} = \left[\phi_{k_i}(p_j)\right]^t, \tag{3.3}$$

for $\phi_{k_i} \in \mathcal{E}_{k_i}$, $i = 1, \dots, r$ and $j = 1, \dots, n$. Furthermore, define the $m_0-$dimensional vector

$$s_{m_0}(q) = \left[\phi_{k_i}(q)\right]^t, \tag{3.4}$$

for $\phi_{k_i} \in \mathcal{E}_{k_i}$, $i = 1, \dots, r$, and let

$$k(q) = \left[K_1(p_1, q), \dots, K_1(p_n, q)\right]^t. \tag{3.5}$$

We have the following.

**Theorem 3.3** *On a compact connected Riemannian manifold $\mathcal{M}$ consider the operator $\mathcal{A} : H_{\mathcal{A}}(\mathcal{M}) \to L^2(\mathcal{M})$ defined by* (2.5). *Suppose $\mathbb{N}_0^{\mathcal{A}}$ is finite,* (2.13) *holds and $p_1, \ldots, p_n$ are distinct points on $\mathcal{M}$, so that* (i) $\{K_0(p_j, q)\}_{j=1}^n$ *spans $H^0_{\mathcal{A}}(\mathcal{M})$, and* (ii) $\{K_1(p_j, q)\}_{j=1}^n$ *is a linearly independent set in $H^1_{\mathcal{A}}(\mathcal{M})$, where $n \geq m_0$ and $q \in \mathcal{M}$. Define*

$$u_{\mathcal{A},n,\xi}(q) = k(q) \cdot c + s(q) \cdot d,$$

*where the $n-$dimensional vector $c$ and the $m_0-$dimensional vector $d$ are defined by*

$$c = K_{n,\xi}^{-1}(I - S_{m_0}X)z, \quad d = Xz, \quad X = \left(S^t_{m_0} K^{-1}_{n,\xi} S_{m_0}\right)^{-1} S^t_{m_0} K^{-1}_{n,\xi},$$

*$z = (z_1, \ldots, z_n)^t$, $z_j \in \mathbb{R}$ and the components of these quantities are defined in* (3.1), (3.2), (3.3), (3.4) *and* (3.5). *Then, $u_{\mathcal{A},n,0}(q)$ is the solution to the interpolation problem, and $u_{\mathcal{A},n,\xi}(q)$ for $\xi > 0$ is the solution to the smoothing problem. Furthermore, $c$ and $d$ are equivalent to*

$$K_{n,\xi} c + S_{m_0} d = z \quad \text{and} \quad S^t_{m_0} c = 0.$$

We once again note that if in addition $\mathcal{M}$ is homogeneous, just as we did in Lemma 2.1, the condition (2.13) in Theorem 3.3 can be replaced by (2.14).

## 3.1 Relationship to the Laplace-Beltrami operator and a proof of Wahba's conjecture

The conditions necessary for solving the interpolation and smoothing problems on $\mathcal{M}$ appear somewhat excessive. This is partly due to the fact that we are looking for spline solutions to general operators $\mathcal{A}$. In the case where $\mathcal{A}$ is the Laplace-Beltrami operator, the conditions appear milder and more familiar. This however, is only apparent for in the case of the Laplace-Beltrami operator, it is found that the conditions of Theorem 3.3 follow from properties of the Laplace-Beltrami operator. Consequently, prior to stating the main result, let us see why this is so.

**Lemma 3.4** *On a compact connected Riemannian manifold $\mathcal{M}$ consider the operator $\mathcal{A}$ : $H_{\mathcal{A}}(\mathcal{M}) \to L^2(\mathcal{M})$ defined by* (2.5) *and suppose $\mathbb{N}_0^{\mathcal{A}}$ is finite. Assume $p_1, \ldots, p_n$ are distinct points on $\mathcal{M}$, $q \in \mathcal{M}$ and suppose $|a_k| \leq \text{const}\lambda_k^{s/2}$ for some positive constant, $s > m/2$ and $k = 0, 1, \ldots$. Then $\{K(p_j, q)\}_{j=1}^n$ is a linearly independent set in $H_{\mathcal{A}}(\mathcal{M})$.*

Thus the importance of this lemma is the positive definiteness of $K$ when $\mathcal{A} = \Delta^{s/2}$.

Consider, the zeta function

$$Z(x, \tilde{s}) = \sum_{k=1}^{\infty} \sum_{\mathcal{E}_k} \lambda_k^{-\tilde{s}} |\phi_k(x)|^2, \tag{3.6}$$

for $\tilde{s} \in \mathbb{C}$, the complex plane. It is known that the series is absolutely convergent in the complex plane when $\mathrm{Re}(\tilde{s}) > m/2$, see Minakshisundaram and Pleijel (1949), Duistermaat and Guillemin (1975), or Hendriks (1990). In fact, define

$$D(x, s, T) = \frac{(2\sqrt{\pi})^m \Gamma(m/2+1)}{\mathrm{vol}\,(\mathcal{M})\, m/2} (s - m/2) T^{s-m/2} \sum_{k=T}^{\infty} \sum_{\mathcal{E}_k} \lambda_k^{-s} |\phi_k(x)|^2,$$

for $x \in \mathcal{M}$ and $s > m/2$. Then it is known that

$$\lim_{T\to\infty} \sup_{x\in\mathcal{M}} \sup_{s>m/2} \left| \frac{D(x, s, T) - 1}{s} \right| = 0,$$

hence the asymptotic behaviour for real $s$ of the zeta function is known, see Theorem 3.1, Hendriks (1990).

For us, the behaviour of the zeta function implies that when the operator in question is the Laplace-Beltrami operator, then condition (2.13) is automatically satisfied whenever $s > m/2$. In the case where $\mathcal{M} = S^1$, the circle, (3.6) would be $\pi^{-1}\zeta(2\tilde{s})$, where $\zeta(\cdot)$ is the Riemann zeta function whose location of zeros is a famous open problem in analytic number theory. In fact, the role of the Riemann zeta function for univariate splines have been noticed in the literature, see Messer (1991, 825).

We therefore summarize the above discussion of the zeta function with the following.

**Lemma 3.5** *On a compact connected Riemannian manifold* $\mathcal{M}$*, consider* $\Delta^{s/2} : H_s(\mathcal{M}) \to L^2(\mathcal{M})$*, where* $s > m/2$*. Then,* (2.13) *is satisfied.* □

This lays the ground work for the main result involving the Laplace-Beltrami operator. Indeed, for $\mathcal{A} = \Delta^{s/2}$, $s > m/2$, define $\langle \cdot, \cdot \rangle_{\Delta^{s/2}} = \langle \cdot, \cdot \rangle_s$, $u_{\mathcal{A},n,\xi} = u_{s,n,\xi}$ and as noted earlier, let $H_s(M)$ denote the Sobolev space of order $s$. We now state the main result which is the formal statement of Wahba's conjecture which of course is now Wahba's theorem.

**Theorem 3.6** (Wahba) *On a compact connected Riemannian manifold $\mathcal{M}$ consider the operator $\Delta^{s/2} : H_s(\mathcal{M}) \to L^2(\mathcal{M})$, where $s > m/2$. Assume $p_1, \ldots, p_n$ are distinct points on $\mathcal{M}$ and $q \in \mathcal{M}$. Define*

$$u_{s,n,\xi}(q) = k(q) \cdot c + s(q) \cdot d,$$

*where the $n-$dimensional vector $c$ and the scalar $d$ are defined by*

$$c = K_{n,\xi}^{-1}(I - S_1)z, \quad d = Xz, \quad X = \left(S_1^t K_{n,\xi}^{-1} S_1\right)^{-1} S_1^t K_{n,\xi}^{-1}$$

*and the components of these quantities are defined in* (3.1), (3.2), (3.3), (3.4) *and* (3.5) *with $m_0 = 1$. Then $u_{s,n,0}(q)$ is the solution to the interpolation problem, and $u_{s,n,\xi}(q)$ for $\xi > 0$ is the solution to the smoothing problem. Furthermore, $c$ and $d$ are equivalent to*

$$K_{n,\xi}c + S_1 d = z \quad \text{and} \quad S_1^t c = 0.$$

# 4 Applications

In this section we will discuss specific applications to higher dimensional spheres $S^{m-1}$ as well as $N \times N$ rotation matrices $SO(N)$. Both are examples of homogeneous compact connected Riemannian manifolds. The latter has in addition a group structure which makes it a compact Lie group. The task therefore is to derive the reproducing kernel (2.10) for these two cases whereby Theorem 3.3 can be applied.

## 4.1 $S^{m-1}$ Higher Dimensional Spheres

Although Wahba (1981) looks exclusively at $S^2$, all of Wahba's arguments extends to $S^{m-1}$, for $m \geq 3$. This is accompolished in Taijeron, Gibson and Chandler (1994). We will briefly outline the general case with concrete expressions for higher dimensional spheres along with the addition formula which can be found for example in Vilenkin (1968), Müller (1998), or Xu (1997).

Indeed, for some $\omega = (\omega_1, \ldots, \omega_m)^t \in S^{m-1}$, the $m-1$ spherical coordinates can be represented by:

$$\begin{aligned} \omega_1 &= \sin\theta_{m-1} \cdots \sin\theta_2 \sin\theta_1 \\ \omega_2 &= \sin\theta_{m-1} \cdots \sin\theta_2 \cos\theta_1 \\ &\vdots \\ \omega_{m-1} &= \sin\theta_{m-1} \cos\theta_{m-2} \\ \omega_m &= \cos\theta_{m-1} \end{aligned} \tag{4.1}$$

where $\theta_1 \in [0, 2\pi)$ and $\theta_j \in [0, \pi)$ for $j = 2, \ldots, m-1$. The invariant measure is

$$d\omega = \frac{\Gamma(m/2)}{2\pi^{m/2}} \sin^{m-2}\theta_{m-1} \cdots \sin\theta_2 d\theta_1 \cdots d\theta_{m-1}.$$

Let $C_r^{\mu}(t)$, $t \in [-1, 1]$ be a polynomial of degree $r$ determined by the power series

$$(1 - 2t\alpha + \alpha^2)^{-\mu} = \sum_{r=0}^{\infty} C_r^{\mu}(t)\alpha^r. \tag{4.2}$$

One notices that $C_r^{1/2}(t)$ are the classical Legendre polynomials. Thus for general $\mu$, these polynomials are generalizations of the classical Legendre polynomials and are called the Gegenbauer (ultraspherical) polynomials.

For $k = (k_1, k_2, \ldots, k_{m-2})$, let $\mathcal{K}_\ell = \{\ell \geq k_1 \geq k_2 \geq \cdots \geq k_{m-2} \geq 0\}$. Define

$$Y_k^{\ell,i} = A_k^\ell \prod_{j=1}^{m-2} C_{k_{j-1}-k_j}^{k_j+(m-j-1)/2}(\cos\theta_{m-j})(\sin\theta_{m-j})^{k_j} Y_i^{k_{m-2}}(\sin\theta_1, \cos\theta_1), \tag{4.3}$$

where

$$Y_1^h(\sin\psi, \cos\psi) = C_h^0(\cos\psi), \quad Y_2^h(\sin\psi, \cos\psi) = \sqrt{2}\sin\psi C_{h-1}^1(\cos\psi),$$

$$Y_2^0(\sin\psi, \cos\psi) = 0 \quad \text{and} \quad [A_k^\ell]^2 = \frac{1}{\Gamma(m/2)} \prod_{j=1}^{m-2} \frac{\Gamma(k_j + (m-j+1)/2)}{\Gamma(k_j + (m-j)/2)}.$$

The collection

$$\left\{Y_k^{\ell,i} : k \in \mathcal{K}_\ell, \ell \geq 0, i = 1, 2\right\}, \tag{4.4}$$

are the eigenfunctions of $\Delta$, the Laplace-Beltrami operator on $S^{m-1}$, $m \geq 3$ and

$$\Delta Y_k^{\ell,i} = \lambda_\ell Y_k^{\ell,i},$$

where

$$\lambda_\ell = \ell(\ell + m - 2), \tag{4.5}$$

$\ell > 0$. Thus each $\ell \geq 0$, determines the eigenspace $\mathcal{E}_\ell$, where

$$\dim \mathcal{E}_\ell = \frac{(2\ell + m - 2)(l + m - 3)!}{\ell!(m-2)!}. \tag{4.6}$$

Collectively, (4.4) is called the spherical harmonics for $L^2(S^{m-1})$ and by virtue of the spectral theorem for compact operators, (4.4) forms a complete orthonormal basis.

The corresponding addition formula for functions on $S^{m-1}$ is as follows. For $\omega, \nu \in S^{m-1}$

$$\sum_{k \in \mathcal{K}_\ell} \left[Y_k^{\ell,1}(\omega)Y_k^{\ell,1}(\nu) + Y_k^{\ell,2}(\omega)Y_k^{\ell,2}(\nu)\right] = \frac{\ell + (m-2)/2}{(m-2)/2} C_\ell^{(m-2)/2}(\omega^t \nu). \tag{4.7}$$

Consequently,

$$K_1(\omega,\nu) = \sum_{\ell \notin \mathbb{N}_0^{\mathcal{A}}} |a_\ell|^{-2} \frac{\ell + (m-2)/2}{(m-2)/2} C_\ell^{(m-2)/2}(\omega^t \nu). \tag{4.8}$$

We note that with regard to Lemma 2.2, (4.7) does not identify the zonal function when $m \geq 4$. We can do so by a renormalization.

Consider the Legendre polynomials $P_\ell(m;t)$, $t \in [-1,1]$ as coefficients in the power series expansion

$$\frac{(1-\alpha^2)}{(1+\alpha^2-2\alpha t)^{3/2}} = \sum_{r=0}^{\infty} \dim \mathcal{E}_\ell \alpha^r P_\ell(m;t),$$

for $m \geq 2$. Note that $P_\ell(3;t)$, $t \in [-1,1]$ are the classical Legendre polynomials. The relationship between the Legendre and Gegenbauer polynomials is

$$C_\ell^{(m-2)/2}(t) = \begin{pmatrix} \ell + m - 3 \\ \ell \end{pmatrix} P_\ell(m;t),$$

for $t \in [-1,1]$ and $m \geq 2$, see Müller (1998, 45). Consequently, we can re-express (4.7) as

$$\sum_{k \in \mathcal{K}_\ell} \left[ Y_k^{\ell,1}(\omega) Y_k^{\ell,1}(\nu) + Y_k^{\ell,2}(\omega) Y_k^{\ell,2}(\nu) \right] = \dim \mathcal{E}_\ell P_\ell(m;\omega^t\nu), \tag{4.9}$$

hence we can re-express (4.8) as

$$K_1(\omega,\nu) = \sum_{\ell \notin \mathbb{N}_0^{\mathcal{A}}} |a_\ell|^{-2} \dim \mathcal{E}_\ell P_\ell(m;\omega^t\nu), \tag{4.10}$$

for $\omega, \nu \in S^{m-1}$ and $m \geq 2$. Since (4.10) is invariant under orthogonal transformations, we can set $\omega = (0,\ldots,0,1)^t$ so that (4.10) only depends on the radial coordinate $\theta_{m-1}$.

Consequently, the zonal function on $S^{m-1}$ is

$$\sqrt{\dim \mathcal{E}_\ell} P_\ell(m;z), \tag{4.11}$$

for $z \in [-1,1]$ and $m \geq 2$.

## 4.2 $SO(N)$ Rotation Matrices

For $N = 2k+1$ odd, let

$$\bar{J} \cap I^* = \left\{ j \in \mathbb{Z}^k : j_1 \geq j_2 \geq \cdots \geq j_k \geq 0 \right\}, \tag{4.12}$$

for $N = 2k$ even, let

$$\bar{J} \cap I^* = \left\{ j \in \mathbb{Z}^k : j_1 \geq j_2 \geq \cdots \geq |j_k| \geq 0 \right\}, \tag{4.13}$$

where $\mathbb{Z}$ denotes the set of all integers. One notices that in the even case, an extra set of indices come from the relation $|j_k|$. The particular form of $\bar{J} \cap I^*$, as classified by the Dynkin diagrams, for $SO(N)$ when $N = 2k+1$ reflects the $\mathrm{B}_k$ root structure $k \geq 2$, while $N = 2k$ reflects the $\mathrm{D}_k$ root structure $k \geq 3$; see for example Bröcker and tom Dieck (1985).

Consider $\Delta$ the Laplace-Beltrami operator on $SO(N)$. For $N = 2k+1$, the corresponding eigenvalue is

$$\lambda_j = j_1^2 + \cdots + j_k^2 + (2k-1)j_1 + (2k-3)j_2 + \cdots + j_k \tag{4.14}$$

while for $N = 2k$

$$\lambda_j = j_1^2 + \cdots + j_k^2 + (2k-2)j_1 + (2k-4)j_2 + \cdots + 2j_{k-1}. \tag{4.15}$$

Further details of the eigenstructure of $SO(N)$ are provided in Appendix B in Kim (1998).

The characters of $SO(N)$ can be evaluated through the Weyl character formula and is taken from Gong (1991) pages 122-123, where $\ell_1 = j_1 + k - 1, \ell_2 = j_2 + k - 2, \ldots, \ell_k = j_k$ and $0 \leq \theta_1, \ldots, \theta_k < \pi$. For $N = 2k+1$, the irreducible character for $j = (j_1, \ldots, j_k)$ is

$$\frac{\begin{vmatrix} \sin(\ell_1 + 1/2)\theta_1 & \cdots & \sin(\ell_1 + 1/2)\theta_k \\ \cdots & \cdots & \cdots \\ \sin(\ell_k + 1/2)\theta_1 & \cdots & \sin(\ell_k + 1/2)\theta_k \end{vmatrix}}{\begin{vmatrix} \sin(k - 1/2)\theta_1 & \cdots & \sin(k - 1/2)\theta_k \\ \cdots & \cdots & \cdots \\ \sin(1/2)\theta_1 & \cdots & \sin(1/2)\theta_k \end{vmatrix}} \tag{4.16}$$

where $k \geq 1$.

For $N = 2k$, we have to take cases. For $j = (j_1, \ldots, j_k)$, first assume that $j_k = 0$. Then the irreducible character is

$$\frac{\begin{vmatrix} \cos \ell_1\theta_1 & \cdots & \cos \ell_1\theta_k \\ \cdots & \cdots & \cdots \\ \cos \ell_{k-1}\theta_1 & \cdots & \cos \ell_{k-1}\theta_k \\ 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \cos(k-1)\theta_1 & \cdots & \cos(k-1)\theta_k \\ \cdots & \cdots & \cdots \\ \cos\theta_1 & \cdots & \cos\theta_k \\ 1 & \cdots & 1 \end{vmatrix}} \tag{4.17}$$

where $k \geq 3$. For $j = (j_1, \ldots, j_k)$, assume that $j_k > 0$. Then the irreducible character is

$$\frac{\begin{vmatrix} \cos \ell_1\theta_1 & \cdots & \cos \ell_1\theta_k \\ \cdots & \cdots & \cdots \\ \cos \ell_k\theta_1 & \cdots & \cos \ell_k\theta_k \end{vmatrix} + i^k \begin{vmatrix} \sin \ell_1\theta_1 & \cdots & \sin \ell_1\theta_k \\ \cdots & \cdots & \cdots \\ \sin \ell_k\theta_1 & \cdots & \sin \ell_k\theta_k \end{vmatrix}}{\begin{vmatrix} \cos(k-1)\theta_1 & \cdots & \cos(k-1)\theta_k \\ \cdots & \cdots & \cdots \\ \cos\theta_1 & \cdots & \cos\theta_k \\ 1 & \cdots & 1 \end{vmatrix}} \tag{4.18}$$

where $k \geq 3$. For $j = (j_1, \dots, j_k)$, assume that $j_k < 0$. Then the irreducible character is

$$\frac{\begin{vmatrix} \cos \ell_1\theta_1 & \cdots & \cos \ell_1\theta_k \\ \cdots & \cdots & \cdots \\ \cos \ell_k\theta_1 & \cdots & \cos \ell_k\theta_k \end{vmatrix} - i^k \begin{vmatrix} \sin \ell_1\theta_1 & \cdots & \sin \ell_1\theta_k \\ \cdots & \cdots & \cdots \\ \sin \ell_k\theta_1 & \cdots & \sin \ell_k\theta_k \end{vmatrix}}{\begin{vmatrix} \cos(k-1)\theta_1 & \cdots & \cos(k-1)\theta_k \\ \cdots & \cdots & \cdots \\ \cos\theta_1 & \cdots & \cos\theta_k \\ 1 & \cdots & 1 \end{vmatrix}} \tag{4.19}$$

where $k \geq 3$.

The specialization to $SO(N)$ therefore comes down to the evaluation of Lemma 2.3. Indeed, for $N = 2k+1$, one would substitute (4.16) into Lemma 2.3 along with using (4.14) for the eigenvalues. For $N = 2k$ when $k \geq 3$, one would use (4.15) for the eigenvalues and proceed in cases. Thus for $j \in \bar{J} \cap I^*$ with $j_k = 0$, one would substitute (4.17) into Lemma 2.3. For $j \in \bar{J} \cap I^*$ with $j_k \neq 0$, one would substitute the sum of (4.18) and (4.19) into Lemma 2.3. Notice that the second terms in (4.18) and (4.19) are of opposite sign so that they would offset each other regardless of whether the group character has an imaginary component or not. Consequently, in light of (2.19), $K(p,q)$ is real valued and one can apply Theorem 3.3 to $SO(N)$. As mentioned previously, because the irreducible characters are defined on the maximal torus of the group $\mathcal{G}$, on $SO(N)$, $K(p,q)$ would depend on $k$ variables where $\dim SO(N) = k(2k+1)$ when $N$ is odd and $\dim SO(N) = k(2k-1)$ when $N$ is even.

# 5 Discussion

At a theoretical level, spline interpolation and smoothing is now possible over compact Riemannian manifolds. This generalizes the 2–sphere case of Wahba (1981), as well as the hypersphere case of Taijeron, Gibson and Chandler (1994).

Although Theorem 3.3 achieves this generalization for a general class of operators, while Theorem 3.6 uses the Laplace-Beltrami operator on compact Riemannian manifolds, nevertheless, one is still confronted with the problem of evaluating (2.11), if one wants to numerically implement this procedure for a specific compact Riemannian manifold. Just as in the spherical cases examined in Wahba (1981) and Taijeron, Gibson and Chandler (1994), whatever $\mathcal{M}$ happens to be, aside from the simple case of the circle $\mathcal{M} = S^1$, if one wants to practically implement this spline procedure, one must evaluate the infinite series in question! At this point, such is a formidable challenge and investigations into expressing (2.11) into a numerically convenient form remains to be worked out!

# 6 Proofs to main results

The key to proving the results of Section 3 are contained in Kimeldorf and Wahba (1971) as well as the adaptation of the latter to hyperspheres by Taijeron, Gibson and Chandler (1994).

**Proof of Theorem 3.3.** Assumption (ii) implies the $n \times n$ matrix $[K_1(p_j, p_i)]$ is positive definite and invertible. Thus the $n \times n$ matrix $K_{n,\xi}$, is invertible for all $\xi \geq 0$. Also, $(n\xi)^{-1}I$ is positive definite, when $\xi > 0$. By assumption (i) and (2.9), the image of $S_{m_0}$ is $H^0_{\mathcal{A}}(\mathcal{M})$, and therefore the rank of $S_{m_0}$ is $m_0$. Furthermore, if we define $\pi_0$ and $\pi_1$ as the orthogonal projections of $H_{\mathcal{A}}(\mathcal{M})$ onto $H^0_{\mathcal{A}}(\mathcal{M})$ and $H^1_{\mathcal{A}}(\mathcal{M})$, respectively, then for $j = 1, \ldots, n \geq m_0$, $\pi_0 K(p_j, q) = K_0(p_j, q)$ and $\pi_1 K(p_j, q) = K_1(p_j, q)$.

For $\xi > 0$, apply Lemma 5.1 from Kimmeldorf and Wahba (1971), or Theorem 1.3.1 from Wahba (1990). The result is the solution $u_{\mathcal{A},n,\xi}$ for the smoothing problem and $u_{\mathcal{A},n,0}$ for the interpolation problem.

The relationship among the vectors and matrices are also easy consequences of Kimmeldorf and Wahba (1971). □

In order to prove Lemma 3.4, we need to regularize the problem as in Lemma 2.3 in Taijeron, Gibson and Chandler (1994). For $p \in \mathcal{M}$, let $(\mathcal{O}_p, \psi_p)$ be a chart, *i.e.*, $\mathcal{O}_p \subset \mathcal{M}$ is an open set with $p \in \mathcal{O}_p$ and $\psi_p : \mathcal{O}_p \to \psi_p(\mathcal{O}_p)$ is a diffeomorphism, where $\psi_p(\mathcal{O}_p) \subset \mathbb{R}^m$ is an open subset. Now define

$$f_{\epsilon,p}(q) = \begin{cases} \exp\{-\rho(p,q)/[\epsilon - \rho(p,q)]\} & \text{if } \rho(p,q) \leq \epsilon, \quad \{\rho(p,q) \leq \epsilon\} \subset \mathcal{O}_p \\ 0 & \text{otherwise.} \end{cases} \tag{6.20}$$

Notice that we can shrink the compact support of $f_{\epsilon,p}$ around the compact closure of a small open neighbourhood around $p \in M$ just as we would do in the Euclidean case. This will enable us to regularize the data points $p_1, \ldots, p_n$. We take note of the fact that our definition (6.20) is intrinsic since it is defined through the Riemannian metric. This is slightly different in Taijeron, Gibson and Chandler (1994), for they use the fact that $S^{m-1}$ is embedded in $\mathbb{R}^m$. We are now ready to prove the lemma.

**Proof of Lemma 3.4.** As in the hypothesis, assume $p_1, \ldots, p_n$ are distinct points in $\mathcal{M}$. Let $\rho(\cdot, \cdot)$ be the Riemannian metric, (2.1) on $\mathcal{M}$ and choose $\epsilon$ such that

$$0 < \epsilon < \min_{i \neq j} \rho(p_i, q_j)/2,$$

for $i, j = 1, \ldots, n$.

Define

$$u_i(q) = f_{\epsilon, p_i}(q), \tag{6.21}$$

where $u_i \in C^\infty(\mathcal{M})$ and $\mathcal{A}u_i \in L^2(\mathcal{M})$ for $i = 1, \ldots, n$ and $q \in M$. We note that $u_i(p_j) = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta for $i, j = 1, \ldots, n$.

Consider the linear combination

$$\sum_{j=1}^n \alpha_j K(p_j, \cdot) = 0. \tag{6.22}$$

Using the fact that $H_s(\mathcal{M})$ is an *rkhs* with reproducing kernel $K$, we note that

$$\langle K(p_j, \cdot), u_i(\cdot) \rangle_s = u_i(p_j), \tag{6.23}$$

for $i, j = 1, \ldots, n$. By applying (6.23) to (6.22), we get

$$0 = \sum_{j=1}^n \alpha_j \langle K(p_j, \cdot), u_i(\cdot) \rangle_s = \sum_{j=1}^n \alpha_j u_i(p_j) = \alpha_i,$$

for $i = 1, \ldots, n$. Thus the lemma follows. □

**Proof of Theorem 3.6.** In the situation where the operator is $\Delta^{s/2}$, $s > m/2$, most of the conditions that are being assumed in Theorem 3.3 automatically follow. In particular, the only eigenspace annihilated by $\Delta^{s/2}$ is $\mathcal{E}_0$, hence $\mathbb{N}_0^{\mathcal{A}} = \{0\}$ and therefore $m_0 = 1$. Consequently, $\{K_0(p_j, q) = 1 : j = 1, \ldots, n\}$ clearly spans $H_s^0(\mathcal{M}) = \mathcal{E}_0$. Furthermore, by Lemma 3.5, (2.13) is fulfilled. As an aside, if $\mathcal{M}$ is homogeneous, then (2.13) is (2.14) and the latter follows from (2.2) if and only if $s > m/2$, see Giné (1975b). By Lemma 3.4, $\{K(p_j, \cdot)\}_{j=1}^n$ is linearly independent in $H_s(\mathcal{M})$, so the only thing remaining is to show that $\{K_1(p_j, \cdot)\}_{j=1}^n$ is a linearly independent set in $H_s^1(\mathcal{M})$. The theorem will then follow as a consequence of Theorem 3.3.

Thus, assume

$$\sum_{j=1}^n \alpha_j K_1(p_j, \cdot) = 0. \tag{6.24}$$

Then

$$\begin{aligned} \sum_{j=1}^n \alpha_j K(p_j, \cdot) &= \sum_{j=1}^n \alpha_j K_0(p_j, \cdot) + \sum_{j=1}^n \alpha_j K_1(p_j, \cdot) \\ &= \sum_{j=1}^n \alpha_j \\ &= a_n, \end{aligned} \tag{6.25}$$

say, because $K_0(p_j, \cdot) = 1$ for $j = 1, \ldots, n$ and by (6.24).

Using the fact that $H_s(\mathcal{M})$ is an *rkhs* with reproducing kernel $K$, along with (6.23), if we apply $\langle \cdot, u_i(\cdot) \rangle_s$ to the beginning and the end of (6.25), we have that $\alpha_i = a_n \langle 1, u_i \rangle_s$ for $i = 1, \ldots, n$. Therefore,

$$a_n \left( 1 - \sum_{j=1}^{n} \alpha_j \langle 1, u_j \rangle_s \right) = 0. \tag{6.26}$$

Now according to (6.20), by continuity and compactness, we know that there exists a constant $M_i \geq 0$ so that

$$\exp \left\{ \frac{-\rho(p_i, q)}{\epsilon - \rho(p_i, q)} \right\} \leq M_i, \quad \text{for} \quad \rho(p_i, q) \leq \epsilon,$$

$i = 1, \ldots, n$.

Therefore,

$$\int_{\{\rho(p_i,q) \leq \epsilon\} \cap \mathcal{O}_{p_i}} u_i(q) dq \leq M_i \operatorname{vol} \left( \{\rho(p_i, q) \leq \epsilon\} \cap \mathcal{O}_{p_i} \right),$$

for $i = 1, \ldots, n$. Consequently,

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i \langle 1, u_i \rangle_s &= \sum_{i=1}^{n} \int_{\{\rho(p_i,q) \leq \epsilon\} \cap \mathcal{O}_{p_i}} u_i(q) dq \\ &\leq n \max_i M_i \operatorname{vol} \left( \{\rho(p_i, q) \leq \epsilon\} \cap \mathcal{O}_{p_i} \right). \end{aligned}$$

Furthermore,

$$1 - \sum_{j=1}^{n} \alpha_j \langle 1, u_j \rangle_s \geq 1 - n \max_i M_i \operatorname{vol} \left( \{\rho(p_i, q) \leq \epsilon\} \cap \mathcal{O}_{p_i} \right),$$

and so by letting $\epsilon \to 0$, we can make the second of the above expression as close to 1 as possible. The consequence is that the term in the parentheses of (6.26) can be made arbitrary close to 1 by choosing $\epsilon > 0$ arbitrarily small. Therefore in order for (6.26) to hold, $a_n = 0$. But if that's the case, then (6.25) would be 0, hence $\alpha_j = 0$ for $j = 1, \ldots, n$ by Lemma 3.4. □

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