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SPLINE INTERPOLATION AND SMOOTHING ON THE SPHERE*

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Abstract. We extend the notion of periodic polynomial splines on the circle and thin plate splines on Euclidean d-space to splines on the sphere which are invariant under arbitrary rotations of the coordinate system. We solve the following problem: Find $u \in \mathcal{H}_m(S)$, a suitably defined reproducing kernel (Sobolev) space on the sphere S to, A) minimize $J_m(u)$ subject to $u(P_i) = z_i$, $i = 1, 2, \cdots, n$, and B) minimize

$$\frac{1}{n}\sum_{i=1}^{n}(u(P_i)-z_i)^2+\lambda J_m(u),$$

where

$$J_m(u) = \int_0^{2\pi}\int_0^{\pi}(\Delta^{m/2}u(\theta,\phi))^2\sin\theta\, d\theta\, d\phi, \qquad m \text{ even}$$

$$= \int_0^{2\pi}\int_0^{\pi}\left\{\frac{(\Delta^{(m-1)/2}u)_\phi^2}{\sin^2\theta}+(\Delta^{(m-1)/2}u)_\theta^2\right\}\sin\theta\, d\theta\, d\phi, \qquad m \text{ odd}.$$

Here Δ is the Laplace–Beltrami operator on the sphere and $J_m(u)$ is the natural analogue on the sphere, of the quadratic functional $\int_0^{2\pi}(u^{(m)}(\theta))^2\, d\theta$ on the circle, which appears in the definition of periodic polynomial splines. $J_m(u)$ may also be considered to be the analogue of

$$\sum_{j=0}^{m}\binom{m}{j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^m u}{\partial x^j\,\partial y^{m-j}}\right)^2 dx\, dy$$

appearing in the definition of thin plate splines on the plane. The solution splines are obtained in the form of infinite series, which do not appear to be convenient for certain kinds of computation. We then replace J_m in A) and B) by a quadratic functional Q_m which is topologically equivalent to J_m on $\mathcal{H}_m(S)$ and obtain closed form solutions to the modified problems which are suitable for numerical calculation, thus providing practical pseudo-spline solutions to interpolation and smoothing problems on the sphere. Convergence rates of the splines and pseudo-splines will be the same. A number of results established or conjectured for polynomial and thin plate splines can be extended to the splines and pseudo-splines constructed here.

Key words. splines on the sphere, spherical harmonics, smoothing on the sphere

1. Introduction. This work is motivated by the following problem. The 500 millibar height (the height above sea level at which the pressure is 500 millibars) is measured (with error) at a large number n of weather stations distributed around the world. It is desired to find a smooth function $u = u(\theta, \phi)$ defined on the surface of the earth (θ = latitude, ϕ = longitude) which is an estimate of the 500 millibar height at position (θ, ϕ). There are many ways that this can be done. In this paper we develop what appears to be the natural generalization to the sphere of periodic interpolating and smoothing splines on the circle (see Golomb [12], Wahba [27]) and thin plate splines on Euclidean d-space (see Duchon [6], Meinguet [18], Wahba [28]).

To obtain a periodic interpolating or smoothing spline on the circle C one seeks the solution to one of the problems: Find $u \in \mathcal{H}_m(C)$ to minimize

A) $\quad J_m(u)$ subject to $u(t_i) = z_i, \quad i = 1, 2, \cdots, n$

or

B) $$\frac{1}{n}\sum_{i=1}^{n}(u(t_i)-z_i)^2+\lambda J_m(u).$$

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Here

$$J_m(u) = \int_0^{2\pi} (u^{(m)}(t))^2\, dt, \tag{1.1}$$

$t_i \in [0, 2\pi]$ and $\mathcal{H}_m(C) = \{u: u, u', \cdots, u^{(m-1)}$ abs. cont., $u^{(m)} \in \mathcal{L}_2[0, 2\pi]$, $u^{(j)}(0) = u^{(j)}(2\pi)$, $j = 0, 1, \cdots, m-1\}$. To find a thin plate interpolating or smoothing spline on Euclidean d-space E^d, one finds $u \in \mathcal{H}_m(E^d)$ to minimize A) or B) above, where now $t_i = (x_{1i}, x_{2i}, \cdots, x_{di}) \in E^d$ and

$$J_m(u) = \sum_{i_1, i_2, \cdots, i_m = 1}^{d} \int_{E^d} \left(\frac{\partial^m u}{\partial x_{i_1}\, \partial x_{i_2}, \cdots, \partial x_{i_m}} \right)^2 dx_1\, dx_2 \cdots, dx_d. \tag{1.2}$$

$\mathcal{H}_m(E^d)$ is defined in Meinguet [18]. To obtain a thin plate interpolating or smoothing spline it is necessary that $2m - d > 0$, since otherwise the evaluation functionals $u \to u(t_i)$ will not be bounded in $\mathcal{H}_m(E^d)$ and thus will not have representers which are used in the construction of the solution.

Duchon has called the solutions to problems involving J_m in Euclidean d-space thin plate splines, because, in two dimensions with $m = 2$,

$$J_2(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2 \left(\frac{\partial^2 u}{\partial x_1\, \partial x_2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right] dx_1\, dx_2$$

is the bending energy of a thin plate. Interpolating and smoothing thin plate splines have been computed in a number of examples by Franke, Utreras, Wahba, Wahba and Wendelberger, and Wendelberger for data given in the form of an analytic function which is evaluated by computer at $t_1, t_2, \cdots, t_n$ [9], for function data with simulated errors [26], [28], [30] and for measured 500 millibar height data [31], with very satisfying results. Fisher and Jerome in a classic early paper [8] answered some important questions concerning interpolation problems on Ω a bounded set in R^d associated with general elliptic operators.

For the analysis of meteorological data, we would like to be able to compute smoothing splines on the sphere. To motivate the definition of J_m for the sphere, we first take a look at the Sobolev spaces $\mathcal{H}_m(C)$ of periodic functions on the circle. $\mathcal{H}_m(C)$ is the collection of square integrable functions u on $[0, 2\pi]$ which satisfy

$$a_0^2 + \sum_{\nu=1}^{\infty} \nu^{2m} a_\nu^2 + \sum_{\nu=1}^{\infty} \nu^{2m} b_\nu^2 < \infty, \tag{1.3}$$

where

$$a_\nu = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \cos \nu\theta u(\theta)\, d\theta, \qquad \nu = 0, 1, \cdots,$$

$$b_\nu = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \sin \nu\theta u(\theta)\, d\theta, \qquad \nu = 1, 2, \cdots.$$

We have

$$J_m(u) \doteq \int_0^{2\pi} (u^{(m)}(\theta))^2\, d\theta = \sum_{\nu=1}^{\infty} \nu^{2m} a_\nu^2 + \sum_{\nu=1}^{\infty} \nu^{2m} b_\nu^2 \tag{1.4}$$

for $u \in \mathcal{H}_m(C)$. $\mathcal{H}_m(C)$ is thus a space of (periodic, square integrable) functions whose Fourier coefficients $\{a_\nu, b_\nu\}$ decay sufficiently fast to satisfy (1.3). The functions $\{\cos \nu\theta, \sin \nu\theta\}$ are the (periodic) eigenfunctions of the operator D^{2m} $(D^{2m}u = u^{(2m)})$

which appears when $J_m(u)$ of (1.1) is integrated by parts and u is sufficiently smooth and periodic:

$$J_m(u) = \int_0^{2\pi} u \cdot D^{2m} u \, d\theta.$$

If one formally integrates (1.2) by parts, and u is sufficiently smooth and decreases to 0 at infinity, then one obtains

$$J_m(u) = (-1)^m \int \cdots \int_{E^d} u \cdot \tilde{\Delta}^m u \, dx_1 \cdots dx_d,$$

where $\tilde{\Delta} u$ is the Laplacian,

$$\tilde{\Delta} u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}.$$

The analogue of $\tilde{\Delta}$ on the sphere is the Laplace–Beltrami operator defined by

$$\Delta u = \frac{1}{\sin^2 \theta} u_{\phi\phi} + \frac{1}{\sin \theta} (\sin \theta u_\theta)_\theta,$$

where $\theta \in [0, \pi]$ is latitude and $\phi \in [0, 2\pi]$ is longitude. This is the restriction of the Laplacian in 3-space to the surface of the sphere; see Courant and Hilbert [3, Chapt. V, VII], and Whittaker and Watson [32]. The role of the eigenfunctions $\{(1/\sqrt{\pi}) \cos \nu\theta, (1/\sqrt{\pi}) \sin \nu\theta\}$ in $\mathcal{H}_m(C)$ is played in $\mathcal{H}_m(S)$, (S is the sphere) by the normalized spherical harmonics $\{Y_\nu^k(\theta, \phi)\}_{\nu=0}^{\infty}\,{}_{k=-\nu}^{\nu}$ (defined in § 2), which are the (periodic) eigenfunctions of the Laplace–Beltrami operator Δ^m, and the role of the eigenvalues $\{\nu^{2m}, \nu^{2m}\}_{\nu=1}^{\infty}$ of D^{2m} is played by the eigenvalues of Δ^m. Δ^m has the single square integrable periodic eigenfunction $Y_0^0(\theta, \phi) = 1$, corresponding to the eigenvalue 0. We now define $\mathcal{H}_m(S)$ as the space of square integrable functions u on S with

$$|u_{00}| < \infty, \qquad \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\lambda_{\nu k}} < \infty, \tag{1.5}$$

where

$$u_{\nu k} = \int_S Y_\nu^k(P) u(P) \, dP \tag{1.6}$$

and $\{\lambda_{\nu k}^{-1}\}$, $(\lambda_{\nu k}^{-1} = [\nu(\nu+1)]^k)$ are the eigenvalues of Δ^m corresponding to $\{Y_\nu^k\}$. $\mathcal{H}_m(S)$ is thus a space of square integrable functions whose Fourier Bessel coefficients with respect to the spherical harmonics decay sufficiently fast to satisfy (1.5). Let J_m be defined by

$$(1.7)\quad \begin{aligned} J_m(u) &= \int_0^{2\pi} \int_0^{\pi} (\Delta^{m/2} u)^2 \sin \theta \, d\theta \, d\phi, \qquad m \text{ even}, \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{(\Delta^{(m-1)/2} u)_\phi^2}{\sin^2 \theta} + (\Delta^{(m-1)/2} u)_\theta^2 \sin \theta \, d\theta \, d\phi, \qquad m \text{ odd}. \end{aligned}$$

It is not hard to show that, for $u \in \mathcal{H}_m(S)$,

$$J_m(u) = \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\lambda_{\nu k}}. \tag{1.8}$$

A number of results which are known or conjectured for polynomial splines on the circle and thin plate splines on E^d will carry over to the thin plate splines and pseudo-splines on the sphere. They include optimality properties of the generalized cross-validation estimate of λ and m [4], [25], convergence rates for smoothing splines with noisy data, properties of associated orthogonal series density estimates, and interpretation of interpolating and smoothing splines as Bayes estimates when u is modeled as the solution to the stochastic differential equation $\Delta^{m/2}u =$ "white noise." Details and further references may be found in Wahba [29]. The corresponding splines when u is modeled as a general stationary autoregressive moving average process on S are also given in Wahba [29], as well as possible models encompassing nonstationarity (anisotropy). The reader interested in meteorological applications may be interested in consulting Stanford [24], where ensembles of $\{u_{\nu k}^2\}$ defined in (1.6) have been computed from measured satellite radiance data and are suggestive of an appropriate choice of m in certain meteorological applications. The results here also show that variational techniques for meteorological data analysis similar to these pioneered by Sasaki [21] and others can be carried out on the sphere; see also Wahba and Wendelberger [30]. Part of the importance of the present work is in its potential applicability to important meteorological problems, some of which are mentioned near the end of § 2.

We seek $u \in \mathcal{H}_m(S)$ to minimize

$$\text{A)} \qquad J_m(u) \quad \text{subject to } u(P_i) = z_i, \qquad i = 1, 2, \cdots, n,$$

$$\text{B)} \qquad \frac{1}{n}\sum_{i=1}^{n}(u(P_i) - z_i)^2 + \lambda J_m(u),$$

where $P_i \in S$, and J_m is defined by (1.7).

We cannot solve these problems for $m = 1$ for the same reason they cannot be solved in E^d for $2m - d \leqq 0$, that is, because the evaluation functions are not continuous in $\mathcal{H}_1(S)$, that is, $\mathcal{H}_1(S)$ is not a reproducing kernel space. However, for $m = 2, 3, \cdots$ we will give the explicit solution to those two problems, which we will call thin plate splines on the sphere. It is actually not hard to obtain the solutions, since we can construct a reproducing kernel for $\mathcal{H}_m(S)$, with $J_m(\cdot)$ as a seminorm, from the well-known eigenfunctions and eigenvalues of the Laplace–Beltrami operator. Given the reproducing kernel [2], the solutions to such problems are well known, and in fact problems A) and B) can be solved with $u(P_i)$ replaced by $L_i u$, where L_i is any continuous linear functional on $\mathcal{H}_m(S)$. See, for example, Kimeldorf and Wahba [17] and references cited there.

Unfortunately we only know the aforementioned reproducing kernels in the form of infinite series. It appears that no closed form expression exists which is convenient for computational purposes. Wendelberger [31] has computed the reproducing kernels given below for m from 2 to 10 by evaluating the infinite series, and it is likely that satisfactory computational procedures for interpolation and smoothing splines on the sphere can be developed based on the infinite series. However, for general continuous linear functionals it may be important to have a reproducing kernel in closed form. Furthermore, to compute certain functionals of the solution, for example, derivatives, it may be important to have a closed form solution. For this reason we suggest replacing $J_m(\cdot)$ by another quadratic functional $Q_m(\cdot)$ which is topologically equivalent to $J_m(\cdot)$ in the sense that there exist α and β, $0 < \alpha < \beta < \infty$ such that

$$\alpha J_m(u) \leqq Q_m(u) \leqq \beta J_m(u), \quad \text{all } u \in \mathcal{H}_m(S).$$

We give the reproducing kernel associated with Q_m in closed form. It involves only logarithms and powers of monomials of sines and cosines, and appears quite suitable for the numerical computation of the solutions of A) and B) and related problems with J_m replaced by Q_m. We will call the resulting interpolating and smoothing functions thin plate pseudo-splines on the sphere. Convergence rates for the thin plate pseudo-splines will be the same as those for the thin plate splines on the sphere because of the topological equivalence of J_m and Q_m.

In § 2 we derive the thin plate spline solutions to problems A) and B); and in § 3 we obtain the thin plate pseudo-spline solutions, where J_m is replaced by Q_m.

We remark that the development of § 2 can no doubt be generalized to establish splines associated with the Laplace–Beltrami operator on compact Riemannian manifolds other than the circle and the sphere; see Gine [10], Hannan [14], Yaglom [33] and Schoenberg [23]. However this is not pursued further.

2. Spherical harmonics and the solution to problems A) and B) on the sphere. The spherical harmonics $\{U_\nu^k(\theta, \phi)\}$ are defined by

$$\begin{aligned} U_\nu^k(\theta,\phi) &= \cos k\phi P_\nu^k(\cos\theta), \qquad k=1,2,\cdots,\nu \\ &= \sin k\phi P_\nu^k(\cos\theta), \qquad k=-1,-2,\cdots,-\nu \\ &= P_\nu(\cos\theta), \qquad k=0, \quad \nu=0,1,2,\cdots, \end{aligned}$$

where $P_\nu^k(z)$ are the Legendre functions of the kth order,

$$P_\nu^k(z) = (1-z^2)^{k/2}\left(\frac{d^k}{dz^k}\right)P_\nu(z),$$

and $P_\nu(z)$ is the νth Legendre polynomial. Recursion formulas for generating the P_ν^k may be found in Abramowitz and Stegun [1].

It is well known that the $\{U_\nu^k, k=-\nu,\cdots,\nu, \nu=0,1,\cdots\}$ form an $\mathcal{L}_2(S)$-complete set of eigenfunctions of the Laplace–Beltrami operator of (1.4) satisfying

$$\Delta U_\nu^k = -\nu(\nu+1)U_\nu^k, \qquad k=-\nu,\cdots,\nu, \quad \nu=0,1,\cdots.$$

See Courant and Hilbert [3], Sansone [22]. Let

$$Y_\nu^0 = \sqrt{\frac{2\nu+1}{4\pi}}U_\nu^0, \qquad \nu=0,1,\cdots,$$

$$Y_\nu^k = 2\sqrt{\frac{2\nu+1}{4\pi}\frac{(\nu-k)!}{(\nu+k)!}}U_\nu^k, \qquad k=1,2,\cdots, \quad \nu=1,2,\cdots.$$

Then (Sansone [22, p. 264, 268])

$$\int_S (Y_\nu^k(P))^2\,dP = 1,$$

and we have the addition formula

$$\sum_{k=-\nu}^{\nu} Y_\nu^k(P)Y_\nu^k(P') = \frac{2\nu+1}{4\pi}P_\nu(\cos\gamma(P,P')),$$

where $\gamma(P, P')$ is the angle between P and P'. The $\{Y_\nu^k\}$ form an orthonormal basis for $\mathcal{L}_2(S)$. Jones [16] has used a finite set of spherical harmonics to estimate 500 millibar heights by regression methods. The spherical harmonics are also utilized in several numerical weather prediction models [11].

Let $\mathscr{H}_m^0(S)$ be the subset of $\mathscr{L}_2(S)$ with an expansion of the form

(2.1)
$$u(P) \sim \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} u_{\nu k} Y_\nu^k(P),$$

where

$$u_{\nu k} = \int_S u(P) Y_\nu^k(P)\, dP,$$

satisfying

$$\sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\lambda_{\nu k}} < \infty,$$

where

(2.2)
$$\lambda_{\nu k} = [\nu(\nu+1)]^{-m}.$$

Functions in $\mathscr{H}_m^0(S)$ satisfy

$$\int_S u(P)\, dP = 0,$$

since the 0, 0th term $Y_0^0 \equiv 1$ has been omitted from the expansion (2.1).
$\mathscr{H}_m^0(S)$ is clearly a Hilbert space with the norm defined by

(2.3)
$$\|u\|_m^2 = \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\lambda_{\nu k}}$$

for any $m \geqq 0$. For $m > 1$, define $K(P, P')$, $(P, P') \in S \times S$ by

(2.4)
$$\begin{aligned} K(P, P') = K_m(P, P') &= \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \lambda_{\nu k} Y_\nu^k(P) Y_\nu^k(P') \\ &\equiv \frac{1}{4\pi} \sum_{\nu=1}^{\infty} \frac{2\nu+1}{\nu^m(\nu+1)^m} P_\nu(\cos \gamma(P, P')). \end{aligned}$$

Since $|P_\nu(z)| \leqq 1$ for $|z| \leqq 1$ (Sansone [22, p. 187]), the series converges uniformly for any $m > 1$ and $K(P, P')$ is a well-defined positive definite function on $S \times S$ with

$$\langle K(P, \cdot), K(P', \cdot)\rangle_m = K(P, P'),$$

where $\langle \cdot, \cdot \rangle_m$ is the inner product induced by (2.3). Furthermore, it is easily verified that, for m an integer >1,

$$\int_S K(P, R) \Delta_{(R)}^m K(P', R)\, dR = K(P, P'),$$

where $\Delta_{(R)}^m$ means the operator Δ^m applied to the variable R. This follows since $\Delta^m Y_\nu^k \equiv \lambda_{\nu k}^{-1} Y_\nu^k$. Thus, for m an integer >1, $K(\cdot, \cdot)$ reproduces under the inner product induced by the norm $J_m^{1/2}(\cdot)$, and

$$J_m(u) = \sum_{\nu=1}^{\infty} \sum_{k=-\nu}^{\nu} \frac{u_{\nu k}^2}{\lambda_{\nu k}},$$

with

$$u_{\nu k} = \int_S u(P) Y_\nu^k(P)\, dP \quad \text{for any } u \in \mathscr{H}_m^0.$$

$\mathscr{H}_m^0(S)$ is therefore the reproducing kernel Hilbert space ($r\,k\,h\,s$) with reproducing kernel (rk)$K(\cdot, \cdot)$.

The space $\mathcal{H}_m(S)$ in which one wants to solve problems A) and B) is

$$\mathcal{H}_m(S)=\mathcal{H}_m^0(S)\oplus\{1\},$$

where $\{1\}$ is the one-dimensional space of constant functions. $\mathcal{H}_m^0(S)$ and $\{1\}$ will be orthogonal subspaces in $\mathcal{H}_m(S)$ if we endow $\mathcal{H}_m(S)$ with the norm defined by

$$\|u\|^2=J_m(u)+\frac{1}{4\pi}\left(\int_S u(P)\,dP\right)^2.$$

The following theorem is an immediate consequence of these facts and Kimeldorf and Wahba [17, Lemmas 3.1, 5.1].

THEOREM 1. *The solutions $u_{n,m}$ and $u_{n,m,\lambda}$ to problems* A) *and* B) *on the sphere are given by*

$$u_{n,m,\lambda}(P)=\sum_{i=1}^{n} c_iK(P,P_i)+d, \tag{2.5}$$

where $\mathbf{c}=(c_i,\cdots,c_n)'$ and d are given by

$$\mathbf{c}=(K_n+n\lambda I)^{-1}[I-T(T'(K_n+n\lambda I)^{-1}T)^{-1}T'(K_n+n\lambda I)^{-1}]\mathbf{z}, \tag{2.6}$$

$$d=(T'(K_n+n\lambda I)^{-1}T)^{-1}T'(K_n+n\lambda I)^{-1}\mathbf{z}, \tag{2.7}$$

*where K_n is the $n\times n$ matrix with j, k*th *entry $(K_n)_{ij}$ given by*

$$(K_n)_{ij}=K(P_i,P_j), \tag{2.8}$$

$$T=(1,\cdots,1)' \tag{2.9}$$

and

$$\mathbf{z}=(z_1,\cdots,z_n)'.$$

Also

$$u_{n,m}\equiv u_{n,m,0}.$$

The continuous linear functionals $L_iu=u(P_i)$ may be replaced in the problem statements by any set of n linearly independent continuous linear functionals on $\mathcal{H}_m(S)$ which are not all identically 0 on $\{1\}$. Then, as is usual in *rk* theory, to obtain the solution one replaces $K(P,P_i)$ in (2.5) by $L_iK(P,\cdot)$, $K(P_i,P_j)$ in (2.8), by $L_{i(P)}L_{j(P')}K(P,P')$, and the ith component of T in (2.9) by $L_i(1)$. (See Kimeldorf and Wahba [17].) One example of useful L_i is $L_iu=\int_{S_i}u(P)\,dP$; i.e., the data functionals are regional averages (see Dyn and Wahba [5]). Furthermore, if $z_i=L_iu+\varepsilon_i$, where u is fixed, unknown function in $\mathcal{H}_m(S)$ and the $\{\varepsilon_i\}$ can be modeled as i.i.d. $\mathcal{N}(0,\sigma^2)$ random variables, then (provided λ is chosen properly; see [4], [30]) an estimate of Lu for L any continuous linear functional on $\mathcal{H}_m(S)$ is provided by $Lu_{m,n,\lambda}$. $Lu=\sin\theta u_\theta(P)$ and $Lu=u_\phi(P)$ are continuous linear functionals on $\mathcal{H}_m(S)$ for $m\geqq 3$. Therefore, this provides a technique for estimating meteorological properties of interest involving the derivatives of u, for example, the geostrophic wind; see [30]. Other potential applications are to the estimation of budgets (Johnson and Downey [15]), and the geostrophic vorticity (Haltiner and Martin [13]).

We remark that the equations (2.6) and (2.7) for $\mathbf{c}$ and d can be readily verified to be equivalent to

$$(K_n+n\lambda I)\mathbf{c}+dT=\mathbf{z}, \tag{2.10}$$

$$T'\mathbf{c}=0. \tag{2.11}$$

If we assume K_n and T are given, then (2.10) and (2.11) lend themselves more readily to numerical solution than the computation of (2.6) and (2.7). See Paihua Montes [19], Wahba [28], Wendelberger [31].

In order to have a closed form expression for $K(P_i)$ it is necessary to sum the series

$$k_m(z)=\sum_{\nu=1}^{\infty}\frac{2\nu+1}{\nu^m(\nu+1)^m}P_\nu(z). \tag{2.12}$$

A closed form expression for $m=1$ $(z\neq 0)$ can be obtained but does not interest us here.

To attempt to sum (2.12) for $m=2$, we note that

$$\frac{2\nu+1}{\nu^2(\nu+1)^2}=\frac{1}{\nu^2}-\frac{1}{(\nu+1)^2}=\int_0^1\log h\left(1-\frac{1}{h}\right)h^\nu\,dh,\qquad \nu=1,2,\cdots. \tag{2.13}$$

Using the generating formula for Legendre polynomials (Sansone [22, p. 169]),

$$\sum_{\nu=1}^{\infty}h^\nu P_\nu(z)=(1-2hz+h^2)^{-1/2}-1,\qquad -1<h<1, \tag{2.14}$$

gives

$$k_2(z)=\int_0^1\log h\left(1-\frac{1}{h}\right)\left(\frac{1}{\sqrt{1-2hz+h^2}}-1\right)dh. \tag{2.15}$$

Repeated attempts to integrate this by parts using formulas for indefinite integrals involving expressions of the form $\sqrt{1-2zh+h^2}$, and related integrals to be found in Pierce and Foster [20] and Dwight [7], led us to terms with a closed form expression plus a term involving Dwight [7, formula 731.1] whose right-hand side is an infinite series. This exercise, plus a helpful conversation with R. Askey who suggested that the sum could be reduced to a dilogarithm, convinced us that no readily computable closed form expression was to be found. For this reason we seek to change the problem slightly so that readily computable interpolating and smoothing formulas can be obtained. We do this in the next section.

3. Thin plate pseudo-splines on the sphere. We seek a norm $Q_m^{1/2}(u)$ on $\mathscr{H}_m^0(S)$ which is topologically equivalent to $J_m^{1/2}(u)$ on $\mathscr{H}_m^0(S)$ and for which the reproducing kernel can be obtained in closed form convenient for computation.

Define

$$Q_m(u)=\sum_{\nu=1}^{\infty}\sum_{k=-\nu}^{\nu}\frac{u_{\nu k}^2}{\xi_{\nu k}},\qquad u_{\nu k}=\int_S u(P)Y_\nu^k(P)\,dP,$$

where

$$\xi_{\nu k}=\left[\left(\nu+\frac{1}{2}\right)(\nu+1)(\nu+2)\cdots(\nu+2m-1)\right]^{-1}. \tag{3.1}$$

Since

$$\frac{1}{m^{2m}\xi_{\nu k}}\leq\frac{1}{\lambda_{\nu k}}\leq\frac{1}{\xi_{\nu k}},\qquad \nu=1,2,\cdots,\quad k=-\nu,\cdots,\nu,\quad m=2,3,\cdots,$$

we have

$$\frac{1}{m^{2m}}Q_m(u)\leq J_m(u)\leq Q_m(u),\qquad u\in\mathscr{H}_m^0(S),$$

and, thus the norms $J_m^{1/2}(\cdot)$ and $Q_m^{1/2}(\cdot)$ are topologically equivalent on $\mathscr{H}_m^0(S)$. The reproducing kernel $R(P, P')$ for $\mathscr{H}_m^0(S)$ with norm $Q_m^{1/2}(\cdot)$ is then

$$
\begin{aligned}
R(P,P') = R_m(P,P') &= \sum_{\nu=1}^{\infty}\sum_{k=-\nu}^{\nu} \xi_{\nu k} Y_\nu^k(P) Y_\nu^k(P') \\
&= \frac{1}{2\pi}\sum_{\nu=1}^{\infty}\frac{1}{(\nu+1)(\nu+2)\cdots(\nu+2m-1)} P_\nu(\cos\gamma(P,P')).
\end{aligned}
\tag{3.2}
$$

A closed form expression can be obtained for $R(P, P')$ as follows. Since

$$\frac{1}{r!}\int_0^1 (1-h)^r h^\nu \, dh \equiv \frac{1}{(\nu+1)\cdots(\nu+r+1)}, \qquad r = 0, 1, 2, \cdots,$$

then by using the generating function (2.14) for the Legendre polynomials we have

$$
\begin{aligned}
R(P,P') &= \frac{1}{2\pi}\sum_{\nu=1}^{\infty}\frac{1}{(\nu+1)(\nu+2)\cdots(\nu+2m-1)} P_\nu(z) \\
&= \frac{1}{2\pi}\left[\frac{1}{(2m-2)!} q_{2m-2}(z) - \frac{1}{(2m-1)!}\right],
\end{aligned}
\tag{3.3}
$$

where

$$z = \cos\gamma(P, P')$$

and

$$q_m(z) = \int_0^1 (1-h)^m (1-2hz+h^2)^{-1/2} \, dh, \qquad m = 0, 1, \cdots. \tag{3.4}$$

Formulas for $\int h^m(1-2hz+h^2)^{-1/2}\,dh$, $m = 0, 1, 2$ and recursion formulas for general m in terms of the formulas for $m-1$ and $m-2$ can be found in Pierce and Foster [20, pp. 165, 174, 177, 196]. q_m was obtained by hand for $m = 0, 1, 2$ and 3. In the middle of this dull exercise P. Bjornstad observed that the MACSYMA program at MIT, which could be called from the computer science department at Stanford where this exercise was taking place, could be used to evaluate $q_m(z)$ recursively. He kindly wrote such a program and the results appear in Table 1. Thus, for example, $R(P, P')$ for $m = 2$ involves q_2 and, from the table,

$$q[2] = \frac{A(12W^2 - 4W) - 6CW + 6W + 1}{2},$$

giving

$$q_2(z) = \frac{1}{2}\left\{\ln\left(1+\sqrt{\frac{2}{1-z}}\right)\left[12\left(\frac{1-z}{2}\right)^2 - 4\frac{(1-z)}{2}\right] - 12\left(\frac{1-z}{2}\right)^{3/2} + 6\left(\frac{1-z}{2}\right) + 1\right\}.$$

Note that $q[0]$ which appears in the $m = 1$ case does not lead to a proper rk since $q_0(1)$ is not finite. However, a proper rk exists for any $m > 1$, and the table can be used to define q_{2m-2} for $m = \frac{3}{2}, 2, \frac{5}{2}, \cdots, 6$.

We collect these results in

THEOREM 2. *The solutions $u_{n,m}$ and $u_{n,m,\lambda}$ to the problems: Find $u \in \mathscr{H}_m(S)$ to*

A′) *minimize $Q_m(u)$ subject to $u(P_i) = z_i$, $i = 1, 2, \cdots, n$,*

B′) *minimize $\frac{1}{n}\sum_{i=1}^{n}(u(P_i) - z_i)^2 + \lambda Q_m(u)$,* *are given by*

$$\hat{u}_{n,m} = \hat{u}_{n,m0}, \qquad \hat{u}_{n,m,\lambda}(P) = \sum_{i=1}^{n} c_i R(P, P_i) + d,$$

where $R(P, P')$ *is defined by* (3.3) *and* (3.4) *and* $\mathbf{c}$ *and* d *are determined by*

$$(R_n + n\lambda I)\mathbf{c} - dT = \mathbf{z}, \qquad T'\mathbf{c} = 0,$$

where R_n *is the* $n \times n$ *matrix with* j, kth *entry* $R(P_i, P_j)$ *and* $T = (1, \cdots, 1)'$.

TABLE 1

$$q_m(z) = \int_0^1 (1-h)^m (1 - 2hz + h^2)^{-1/2}\, dh, \qquad m = 0, 1, \cdots, 10,$$

Key. $q[m] = q_m[z]$, $A = \ln(1 + 1/\sqrt{W})$, $C = 2\sqrt{W}), W = (1-z)/2$

$q[0]$	A
$q[1]$	$2AW - C + 1$
$q[2]$	$(A(12W^2 - 4W) - 6CW + 6W + 1)/2$
$q[3]$	$(A(60W^3 - 36W^2) + 30W^2 + C(8W - 30W^2) - 3W + 1)/3$
$q[4]$	$(A(840W^4 - 720W^3 + 72W^2) + 420W^3 + C(220W^2 - 420W^3) - 150W^2 - 4W + 3)/12$
$q[5]$	$(A(7560W^5 - 8400W^4 + 1800W^3) + 3780W^4 + C(-3780W^4 + 2940W^3 - 256W^2) - 2310W^3 + 60W^2 - 5W + 6)/30$
$q[6]$	$(A(27{,}720W^6 - 37{,}800W^5 + 12{,}600W^4 - 600W^3) + 13{,}860W^5 + C(-13{,}860W^5 + 14{,}280W^4 - 2772W^2) - 11970W^4 + 1470W^3 + 15W^2 - 3W + 5)/30$
$q[7]$	$(A(360{,}360W^7 + 582{,}120W^6 + 264{,}600W^5 + 29{,}400W^4) + 180{,}180W^6 + C(-180{,}180W^6 + 231{,}000W^5 - 71{,}316W^4 + 3072W^3) - 200{,}970W^5 + 46{,}830W^4 - 525W^3 + 21W^2 - 7W + 15)/105$
$q[8]$	$(A(10{,}810{,}800W^8 - 20{,}180{,}160W^7 + 11{,}642{,}400W^6 + 2{,}116{,}800W^5 + 58{,}800W^4) + 5{,}405{,}400W^7 + C(-5{,}405{,}400W^7 + 8{,}288{,}280W^6 - 3{,}538{,}920W^6 + 363{,}816W^4) - 7{,}387{,}380W^6 + 2{,}577{,}960W^5 - 159{,}810W^4 - 840W^3 + 84W^2 - 40W + 105)/840$
$q[9]$	$(A(61{,}261{,}200W^9 - 129{,}729{,}600W^8 + 90{,}810{,}720W^7 - 23{,}284{,}800W^6 + 1{,}587{,}600W^5) + 30{,}630{,}600W^8 + C(-30{,}630{,}600W^8 + 54{,}654{,}600W^7 - 29{,}909{,}880W^6 + 5{,}104{,}440W^5 - 131{,}072W^4) - 49{,}549{,}500W^7 + 23{,}183{,}160W^6 - 2{,}903{,}670W^5 + 17{,}640W^4 - 420W^3 + 72W^2 - 45W + 140)/1{,}260$
$q[10]$	$(A(232{,}792{,}560W^{10} - 551{,}350{,}800W^9 + 454{,}053{,}600W^8 - 151{,}351{,}200W^7 + 17{,}463{,}600W^6 - 317{,}520W^5) + 116{,}396{,}280W^9 + C(-116{,}396{,}280W^9 + 236{,}876{,}640W^8 - 158{,}414{,}256W^7 + 38{,}507{,}040W^6 - 2{,}462{,}680W^5) - 217{,}477{,}260W^8 + 127{,}987{,}860W^7 - 24{,}954{,}930W^6 + 930{,}006W^5 + 2{,}940W^4 - 180W^3 + 45W^2 - 35W + 126)/1{,}260$

see errata sheet

Of course the remarks following Theorem 1 concerning general continuous linear functionals and computing procedures apply here also.

4. Acknowledgments. We would like to thank P. Bjornstad for writing the computer program which generated Table 1, G. Golub for his hospitality at the Stanford University Numerical Analysis Group where some of this work was done and M. Ghil for helpful discussions concerning related stochastic differential equations.

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ERRATUM: SPLINE INTERPOLATION AND SMOOTHING ON THE SPHERE*

GRACE WAHBA†

Table 1 contains several misprints in lines q [6], q [7] and q [8]. The correct table appears below.

TABLE 1

$$q_m(z)=\int_0^1 (1-h)^m(1-2hz+h^2)^{-1/2}\,dh, \qquad m=0,1,\cdots,10.$$

Key. $q[m]=q_m(z)$, $A=\ln(1+1/\sqrt{W})$, $C=2\sqrt{W}$, $W=(1-z)/2$

q[0];

$$A$$

q[1];

$$2AW-C+1$$

q[2];

$$\frac{A(12W^2-4W)-6CW+6W+1}{2}$$

q[3];

$$\frac{A(60W^3-36W^2)+30W^2+C(8W-30W^2)-3W+1}{3}$$

q[4];

$$\frac{A(840W^4-720W^3+72W^2)+420W^3+C(220W^2-420W^3)-150W^2-4W+3}{12}$$

q[5];

$$(A(7560W^5-8400W^4+1800W^3)+3780W^4 + C(-3780W^4+2940W^3-256W^2)-2310W^3+60W^2-5W+6)/30$$

q[6];

$$(A(27720W^6-37800W^5+12600W^4-600W^3)+13860W^5 + C(-13860W^5+14280W^4-2772W^3)-11970W^4+1470W^3+15W^2-3W+5)/30$$

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continued

q[7];

$$(A\,(360360\,W^7 - 582120\,W^6 + 264600\,W^5 - 29400\,W^4) + 180180\,W^6 + C\,(-\,180180\,W^6 + 231000\,W^5 - 71316\,W^4 + 3072\,W^3) - 200970\,W^5 + 46830\,W^4 - 525\,W^3 + 21\,W^2 - 7\,W + 15)/105$$

q[8];

$$(A\,(10810800\,W^8 - 20180160\,W^7 + 11642400\,W^6 - 2116800\,W^5 + 58800\,W^4) + 5405400\,W^7 + C\,(-\,5405400\,W^7 + 8288280\,W^6 - 3538920\,W^5 + 363816\,W^4) - 7387380\,W^6 + 2577960\,W^5 - 159810\,W^4 - 840\,W^3 + 84\,W^2 - 40\,W + 105)/840$$

q[9];

$$(A\,(61261200\,W^9 - 129729600\,W^8 + 90810720\,W^7 - 23284800\,W^6 + 1587600\,W^5) + 30630600\,W^8 + C\,(-\,30630600\,W^8 + 54654600\,W^7 - 29909880\,W^6 + 5104440\,W^5 - 131072\,W^4) - 49549500\,W^7 + 23183160\,W^6 - 2903670\,W^5 + 17640\,W^4 - 420\,W^3 + 72\,W^2 - 45\,W + 140)/1260$$

q[10];

$$(A\,(232792560\,W^{10} - 551350800\,W^9 + 454053600\,W^8 - 151351200\,W^7 + 17463600\,W^6 - 317520\,W^5) + 116396280\,W^9 + C\,(-\,116396280\,W^9 + 236876640\,W^8 - 158414256\,W^7 + 38507040\,W^6 - 2462680\,W^5) - 217477260\,W^8 + 127987860\,W^7 - 24954930\,W^6 + 930006\,W^5 + 2940\,W^4 - 180\,W^3 + 45\,W^2 - 35\,W + 126)/1260$$