whence it follows that

$$\|\Phi\| \geqq \|f\|.$$

Thus the theorem of F. Riesz is proved.

We consider now a linear functional Ψ with domain D_Ψ closed in H. Then D_Ψ is a subspace of H and the theorem of F. Riesz asserts the existence of a unique element $g \in D_\Psi$ such that

$$\Psi(h) = (h, g) \qquad (h \in D_\Psi) \tag{3}$$

and

$$\|\Psi\|_{D_\Psi} = \|g\|.$$

By means of (3), the linear functional Ψ may be extended to the whole space H without increasing the norm.[3] Any other extension of the linear functional Ψ to the whole space H increases the norm of the functional. In fact, if Φ is any extension of Ψ to the whole space, then

$$\Phi(h) = (h, f)$$

and

$$\|\Phi\| = \|f\|.$$

For $h \in D_\Psi$,

$$(h, g) = (h, f)$$

so that $f - g \perp D_\Psi$. Because $g \in D_\Psi$,

$$\|f\|^2 = \|g\|^2 + \|f - g\|^2$$

which implies that

$$\|\Phi\| \geqq \|\Psi\|_{D_\Psi},$$

where there is strict inequality if $f \neq g$.

17. A Criterion for the Closure in H of a Given System of Vectors

According to the definition in Section 8, a system M of vectors is closed in H if it is possible to approximate each $h \in H$ to any degree of accuracy by means of a linear combination of vectors belonging to M.

THEOREM: *In order that the system* M *be closed in* H, *it is necessary and sufficient that a linear functional* Φ *in* H *which vanishes for all* $g \in M$, *be identically equal to zero.*

Proof: The necessity is an immediate consequence of the continuity of the linear functional. In order to prove the sufficiency, let us assume that the system is not closed. Then there exists $\delta > 0$ and a vector $h_0 \in H$ for which

$$\inf_{n, \alpha_i} \|h_0 - \alpha_1 g_1 - \alpha_2 g_2 - \dots - \alpha_n g_n\| = \delta > 0 \qquad (g_i \in M).$$

[3] Since any linear functional can be extended to the whole space without increasing the norm, one usually considers a linear functional as being defined on the whole space when the domain is not specified.

We denote by G the closed linear envelope of the system M. On the basis of Section 6, there exists $g \in G$ such that

$$\| h_0 - g \| = \delta.$$

Let

$$f = h_0 - g.$$

Then $f \perp G$. Consider the functional Φ defined by

$$\Phi(h) = (h, f),$$

the norm of which is equal to $\| f \| = \delta > 0$. This nonzero functional vanishes for each vector of G and, in particular, for each vector of M. Thus, the sufficiency is also proved.

18. A Lemma Concerning Convex Functionals[4]

Definition: *A real functional $p(h)$ in* H *is said to be convex if*

$$p(f+g) \leqq p(f) + p(g) \tag{1}$$

and

$$p(\alpha f) = |\alpha| p(f), \tag{2}$$

for $f, g \in$ H and any complex number α.

From this definition it follows that $p(0) = 0$ and $p(h) \geqq 0$.

LEMMA: *If a convex functional $p(h)$ is lower semicontinuous*, i.e., *if for each $h_0 \in$ H and each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$p(h) - p(h_0) > -\varepsilon$$

for $\|h - h_0\| < \delta$, then the convex functional is bounded, i.e., *there exists* $M > 0$ *such that*

$$p(h) \leqq M \| h \|$$

for $h \in$ H.

Proof:[5] First, we prove that if the functional is not bounded in the unit sphere ($\| h \| < 1$), then it will not be bounded in the sphere $S(\rho, g)$ with center $g \in$ H and radius $\rho > 0$, where g and ρ are arbitrary. For, assuming that $p(h) < C$ for $\|h - g\| < \rho$, we find that

$$p(h-g) \leqq p(h) + p(-g) = p(h) + p(g) < 2C$$

for $\|h - g\| < \rho$. Consequently, if

$$f = \frac{h-g}{\rho},$$

[4] In this section we follow I. M. Gelfand [1].

[5] This proof does not require that H be a Hilbert space; it goes through if H is any Banach space.

then

$$p(f) = \frac{2C}{\rho}$$

for $f \in S(1, 0)$, so that the functional $p(h)$ is bounded in the unit sphere. In view of property (2) it is sufficient to prove that the functional $p(h)$ is bounded in the sphere $S(1, 0)$. We assume the contrary. Then $p(h)$ is unbounded in every sphere. We choose a point $f_1 \in S(1, 0)$ such that $p(f_1) > 1$. The lower semicontinuity of the functional $p(h)$ implies that there is a sphere $S(\rho_1, f_1) \subset S(1, 0)$ with radius $\rho_1 < \frac{1}{2}$ at all points of which $p(h) > 1$. Since $p(h)$ is unbounded in every sphere, there exists a point $f_2 \in S(\rho_1, f_1)$ and also a sphere $S(\rho_2, f_2) \subset S(\rho_1, f_1)$ with radius $\rho_2 < \frac{1}{2}\rho_1$ in which $p(h) > 2$. Continuing this process, we get an infinite sequence of spheres,

$$S(1, 0) \supset S(\rho_1, f_1) \supset S(\rho_2, f_2) \supset \dots$$

for which $\rho_n < \frac{1}{2}\rho_{n-1}$, $(n = 1, 2, 3, \dots; \rho_0 = 1)$, and also $p(h) > n$ if $h \in S(\rho_n, f_n)$. But the sequence of centers $\{f_n\}_1^\infty$ is fundamental and, therefore, converges to some element f. Then $p(f) > n$ for each n, which is impossible. Thus, the lemma is proved.

We remark that this lemma can also be formulated as follows: if a convex functional is lower semicontinuous, then it is continuous.

COROLLARY: *Let $p_k(h)$, $(k = 1, 2, 3, \dots)$ be a sequence of convex continuous functionals in* H. *If this sequence is bounded at each point $h \in$ H, then the functional*

$$p(h) = \sup_n p_n(h)$$

is also convex and continuous.

Proof: That $p(h)$ is a convex functional is evident. On the other hand, for each $h_0 \in$ H and each $\varepsilon > 0$, there exists N such that

$$p(h_0) - p_N(h_0) < \frac{\varepsilon}{2}.$$

Then there exists $\delta > 0$ such that

$$|p_N(h) - p_N(h_0)| < \frac{\varepsilon}{2}$$

for $\|h - h_0\| < \delta$. But if $\|h - h_0\| < \delta$ then

$$p(h) - p(h_0) > \sup_n p_n(h) - p_N(h_0) - \frac{\varepsilon}{2} \geq p_N(h) - p_N(h_0) - \frac{\varepsilon}{2} > -\varepsilon.$$

This implies that the functional $p(h)$ is lower semicontinuous. It remains only to apply the lemma, and the corollary is proved.

We give two simple applications of the propositions just proved. We know that each linear functional in $L^2(a, b)$ can be expressed in the form

$$(1) \qquad \Phi(h) = \int_a^b h(t)\,\varphi(t)\,dt$$

where $\varphi(t)$ is the function in $L^2(a, b)$ which "represents" the functional $\Phi(h)$. We shall prove that if a functional $\Phi(h)$ is defined everywhere in $L^2(a, b)$ by means of formula (1), where $\varphi(t)$ is some fixed function, then this functional is necessarily linear, so that $\varphi(t)$ belongs to $L^2(a, b)$. In other words we shall prove that if the integral (1) exists for each function $h(t) \in L^2(a, b)$, then $\varphi(t) \in L^2(a, b)$. This fact is a special case of a more general theorem of F. Riesz.[6]

For the proof we denote by e_n the set of all points t which belong to the intersection of the intervals $[a, b]$, $[-n, n]$ and for which

$$|\varphi(t)| \leqq n.$$

Further let

$$p_n(h) = \int_{e_n} |h(t)\varphi(t)|\,dt.$$

This is a convex continuous functional in $L^2(a, b)$. The quantity

$$p(h) = \sup_n p_n(h) = \lim_{n\to\infty} p_n(h) = \int_a^b |h(t)\varphi(t)|\,dt$$

is finite for any $h(t) \in L^2(a, b)$. So, by the corollary of the lemma, the functional $p(h)$ is continuous, i.e., $p(h) \leqq M \,\|h\|$ for $h \in \mathrm{H}$. But $|\Phi(h)| \leqq p(h)$ so that, since the homogeneity and additivity of the functional $\Phi(h)$ are evident, $\Phi(h)$ is a linear functional.

An analogous proposition is valid for the space l^2. We restrict ourselves to its formulation. Let a functional $\Phi(f)$ be defined everywhere in l^2 by means of the formula

$$\Phi(f) = \sum_{k=1}^{\infty} a_k x_k \qquad (f = \{x_k\}_1^{\infty})$$

where $\{a_k\}_1^{\infty}$ is some fixed sequence. Then

$$(2) \qquad \sum_{k=1}^{\infty} |a_k|^2 < \infty,$$

---

[6] Riesz's theorem pertains to the space $L^p(a, b)$ for any $p > 1$. (The space $L^p(a, b)$ is defined as the space of functions measurable in (a, b) for which $\int_a^b |f(x)|^p dx$ exists). See F. Riesz [1].

which implies that $\Phi(f)$ is a linear functional. In other words, if the series

$$\sum_{k=1}^{\infty} a_k x_k$$

converges for each sequence $\{x_k\}_1^\infty$ such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty$$

then the inequality (2) must hold. This fact is a special case of a more general theorem of E. Landau.[7]

19. Bounded Linear Operators

An operator T is *linear* if its domain of definition D is a linear manifold and if

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg$$

for any $f, g \in \mathrm{D}$ and any complex numbers α and β.

We emphasize the fact that, in contrast with the definition of a linear functional, this definition does not require that the operator be bounded. This is related to the fact that many important operations of analysis such as, for instance, the operation of differentiation, generate unbounded but homogeneous and additive operators, i.e., operators which are linear in the sense of the definition given here.

A linear operator T is *bounded* if

$$\sup_{f \in \mathrm{D}, \|f\| \leq 1} \|Tf\| < \infty.$$

The left member of this inequality is called the *norm* of the operator T in D and is denoted by the symbol $\|T\|$ or, sometimes, by $\|T\|_{\mathrm{D}}$.

It is easy to see that the properties of Section 15 relating to linear functionals are also valid for bounded linear operators:

1. The norm of a bounded linear operator T can be defined equivalently by

$$\|T\| = \sup_{f \in \mathrm{D}, \|f\| = 1} \|Tf\| = \sup_{f \in \mathrm{D}} \frac{\|Tf\|}{\|f\|}.$$

2. A bounded linear operator is continuous.
3. If a linear operator is continuous at one point, then it is bounded.
4. The extension by continuity of a bounded linear operator T leads to a unique linear operator with the same norm as the original operator.

---

[7] Landau's theorem pertains to the space l^p for any $p > 1$. (The space l^p is the space of numerical sequences $x_1, x_2, \ldots$ for which the series $\sum_{i=1}^{\infty} |x_i|^p$ converges.) See E. Landau [1].

5. If S and T are linear operators, then $\alpha S + \beta T$, where α and β are complex numbers, is a linear operator with the intersection $D_S \cap D_T$ of the domains D_S and D_T as the domain of definition. Each of the products ST and TS (cf. Section 14) is also a linear operator. If S and T are bounded linear operators defined everywhere in H, then the operators ST and TS are also bounded linear operators defined everywhere in H, and

$$\| ST \| \leqq \| S \| \cdot \| T \|, \qquad \| TS \| \leqq \| T \| \cdot \| S \|.$$

20. Bilinear Functionals

We shall say that Ω is a bilinear functional defined in H,[8] if to each pair of elements $f, g \in \mathrm{H}$ there corresponds a definite complex number $\Omega(f,g)$, and

(a) $\Omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Omega(f_1, g) + \alpha_2 \Omega(f_2, g)$,

(b) $\Omega(f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \Omega(f, g_1) + \bar{\beta}_2 \Omega(f, g_2)$,

(c) $\sup\limits_{\|f\| \leqq 1,\, \|g\| \leqq 1} | \Omega(f,g) | < \infty.$

An example of a bilinear functional is the scalar product (f, g). The number

$$\sup_{\|f\| \leqq 1,\, \|g\| \leqq 1} | \Omega(f,g) |$$

is called the norm of the bilinear functional Ω, and is denoted by the symbol $\| \Omega \|$. It is not difficult to prove that

$$\| \Omega \| = \sup_{\|f\| = 1,\, \|g\| = 1} |\Omega(f,g)| = \sup \frac{|\Omega(f,g)|}{\|f\| \cdot \|g\|}.$$

Therefore, for any $f, g \in \mathrm{H}$,

$$|\Omega(f,g)| \leqq \| \Omega \| \cdot \|f\| \cdot \|g\|.$$

A bilinear functional is a continuous function of each of its arguments, since

$$|\Omega(f,g) - \Omega(f_0,g_0)| = |\Omega(f-f_0, g-g_0) + \Omega(f-f_0, g_0) + \Omega(f_0, g-g_0)| \leqq$$
$$\leqq \|\Omega\| \{\|f-f_0\| \cdot \|g-g_0\| + \|f-f_0\| \cdot \|g_0\| + \|f_0\| \cdot \|g-g_0\|\}.$$

The following simple proposition is often useful.

THEOREM: *If a complex scalar function $\omega(f, g)$ satisfies the conditions*

(a) $\omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \omega(f_1, g) + \alpha_2 \omega(f_2, g)$,

(b) $\omega(f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \omega(f, g_1) + \bar{\beta}_2 \omega(f, g_2)$,

(c) $| \omega(f,f) | \leqq C \|f\|^2$,

(d) $| \omega(f,g) | = | \omega(g,f) |$,

[8] It is possible to introduce bilinear functionals which are not defined everywhere in H, but in what follows they will not be considered.

where C is a constant, f, f_1, f_2, g, g_1, g_2 are arbitrary elements of H *and α_1, α_2, β_1, β_2 are arbitrary complex numbers, then ω is a bilinear functional with norm $\|\omega\| \leqq C$.*

Proof: It is immediately proved by means of (a) and (b) that[9]

$$\omega(f,h)+\omega(h,f)=\tfrac{1}{2}\{\omega(f+h,f+h)-\omega(f-h,f-h)\}.$$

This implies that

$$(1)\quad |\omega(f,h)+\omega(h,f)|\leqq\tfrac{1}{2}C\{\|f+h\|^2+\|f-h\|^2\}=C\{\|f\|^2+\|h\|^2\}.$$

Let $\|f\| \leqq 1$, $\|h\| \leqq 1$ and $h=\lambda g$ where λ is a complex number such that $|\lambda|=1$ (λ will be specified later). Then (1) yields

$$(2)\qquad |\bar{\lambda}\omega(f,g)+\lambda\omega(g,f)|\leqq 2C.$$

We suppose that $\omega(f,g)\neq 0$ and, in accordance with (d), let

$$\omega(f,g)=|\omega(f,g)|e^{i\alpha},\qquad \omega(g,f)=|\omega(f,g)|e^{i\beta}.$$

Then, by (2),

$$|\omega(f,g)|\cdot|\bar{\lambda}e^{i\alpha}+\lambda e^{i\beta}|\leqq 2C.$$

Letting

$$\lambda=e^{i\frac{\alpha-\beta}{2}},$$

we find that

$$\bar{\lambda}e^{i\alpha}+\lambda e^{i\beta}=e^{i\frac{\alpha+\beta}{2}}+e^{i\frac{\alpha+\beta}{2}}=2e^{i\frac{\alpha+\beta}{2}}$$

which yields

$$|\omega(f,g)|\leqq C\qquad(\|f\|\leqq 1,\|g\|\leqq 1).$$

This proves the theorem, since this relation is also correct for $\omega(f,g)=0$.

COROLLARY: *If the bilinear functional Ω satisfies the condition*

$$|\Omega(f,g)|=|\Omega(g,f)|,$$

for $f,g\in$ H, then

$$\|\Omega\|=\sup_{f\in\mathrm{H}}\frac{|\Omega(f,f)|}{(f,f)}.$$

Proof: By the theorem,

$$\|\Omega\|\leqq\sup_{f\in\mathrm{H}}\frac{|\Omega(f,f)|}{(f,f)},$$

but on the other hand

$$\sup_{f\in\mathrm{H}}\frac{|\Omega(f,f)|}{(f,f)}\leqq\sup_{f,g\in\mathrm{H}}\frac{|\Omega(f,g)|}{\|f\|\cdot\|g\|}=\|\Omega\|.$$

---

[9] From this equation and the analogous equation

$$\omega(f,h)-\omega(h,f)=\frac{i}{2}\{\omega(f+ih,f+ih)-\omega(f-ih,f-ih)\}$$

it follows by means of (c) that ω is a bilinear functional with norm $\leqq 2C$. But, by means of condition (d), it is established that the norm of ω does not exceed C.

21. The General Form of a Bilinear Functional

THEOREM: *Each bilinear functional $\Omega(f, g)$ has a representation of the form*

$$\Omega(f,g) = (Af,g).$$

In this equation A is a bounded linear operator with domain H *which is uniquely determined by Ω. Furthermore,*

$$\|A\| = \|\Omega\|.$$

Proof: For fixed f, the expression $\overline{\Omega(f, g)}$ defines a linear functional in g with domain H. Consequently, according to the theorem of F. Riesz (cf. Section 16), there exists an element h_f, uniquely determined by the element f, for which

$$\overline{\Omega(f,g)} = (g, h_f)$$

or

$$\Omega(f,g) = (h_f, g)$$

for each $g \in$ H. Define the mapping A from H into H by the equation $Af = h_f$ for $f \in$ H. Then

$$\Omega(f,g) = (Af,g).$$

Since

$$\Omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Omega(f_1, g) + \alpha_2 \Omega(f_2, g)$$

we have

$$(A\{\alpha_1 f_1 + \alpha_2 f_2\} - \alpha_1 A f_1 - \alpha_2 A f_2, g) = 0$$

for $g \in$ H. Since g is arbitrary,

$$A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A f_1 + \alpha_2 A f_2,$$

so that A is a linear operator. The domain of the operator A is the whole space H. Furthermore, since

$$|(Af,g)| \leqq \|Af\| \cdot \|g\|$$

we have

$$\|\Omega\| = \sup \frac{|\Omega(f,g)|}{\|f\| \cdot \|g\|} = \sup \frac{|(Af,g)|}{\|f\| \cdot \|g\|} \leqq \sup \frac{\|Af\|}{\|f\|}.$$

On the other hand

$$\|\Omega\| = \sup \frac{|(Af,g)|}{\|f\| \cdot \|g\|} \geqq \sup \frac{(Af,Af)}{\|f\| \cdot \|Af\|} = \sup \frac{\|Af\|}{\|f\|}.$$

These relations show that the operator A is bounded and that

$$\|\Omega\| = \|A\|.$$

The operator A is uniquely determined by the linear functional Ω. In fact, if

$$\Omega(f,g) = (A'f,g) = (A''f,g),$$

for $f, g \in$ H, then

$$(A'f - A''f, g) = 0.$$

But this is possible only for $A' = A''$.

22. Adjoint Operators

Let A be an arbitrary bounded linear operator defined on H. The expression

$$(f, Ag)$$

defines a bilinear functional on H with norm $\| A \|$. According to the theorem proved in the preceding section, there exists a unique bounded linear operator A^* defined on H with norm $\| A^* \| = \| A \|$ such that

$$(1) \qquad (f, Ag) = (A^*f, g)$$

for $f, g \in$ H. This operator A^* is called the *adjoint* of A. It is easy to see that the operator $(A^*)^* = A^{**}$ is equivalent to the original operator A.

If A is bounded and $A^* = A$, then A is said to be *self-adjoint*. A bounded linear operator A, defined on H, is said to be *normal* if it commutes with its adjoint, i.e., if

$$A^*A = AA^*.$$

Let A and B be two bounded linear operators defined on H. Then

$$(ABf, g) = (Bf, A^*g) = (f, B^*A^*g),$$

which implies that

$$(AB)^* = B^*A^*.$$

Therefore, the product of two self-adjoint operators is self-adjoint if and only if the operators commute.

THEOREM: *If A is a bounded self-adjoint operator, then*

$$\sup_{\|f\| = \|g\| = 1} |(Af, g)| = \sup_{\|f\| = 1} |(Af, f)|.$$

In other words,[10]

$$\|A\| = \max \{ |\Lambda|, |\lambda| \}$$

where

$$\Lambda = \sup_{\|f\|=1} (Af, f), \quad \lambda = \inf_{\|f\|=1} (Af, f).$$

Proof: The bilinear functional

$$\Omega(f, g) = (Af, g)$$

satisfies the condition

$$|\Omega(f, g)| = |\Omega(g, f)|.$$

Therefore, the corollary of the theorem of Section 20 applies and the theorem is proved.

[10] Translator's Note: It follows from (1) that (Af, f) is real for $f \in$ H if A is self-adjoint.

23. Weak Convergence in H

We say that the sequence of vectors $f_k \in \mathrm{H}$, $(k = 1, 2, 3 \ldots)$ *converges weakly* to the vector f and we write $f_k \xrightarrow{w} f$ if

$$\lim_{k\to\infty} (f_k, h) = (f, h)$$

for $h \in \mathrm{H}$. The concepts of a *weakly fundamental sequence* and of *weak completeness* are defined analogously.

If the sequence $\{f_k\}_1^\infty$ converges to f in the sense of Section 3, i.e., if

$$\lim_{k\to\infty} \|f_k - f\| = 0$$

then we shall continue to write $f_k \to f$, but we shall say, to avoid confusion, that the sequence converges *strongly* to f. Strong convergence implies weak convergence, but not conversely. Indeed, let $\{e_k\}_1^\infty$ be any infinite orthonormal sequence of vectors in H. Since, for any $h \in \mathrm{H}$,

$$\sum_{k=1}^{\infty} |(h, e_k)|^2 \leqq (h, h)$$

(see Section 8), then for any $h \in \mathrm{H}$,

$$\lim_{k\to\infty} (e_k, h) = 0.$$

Thus, the sequence $\{e_k\}_1^\infty$ converges weakly to the vector 0, but this sequence does not converge strongly since

$$\| e_k - e_i \|^2 = 2 \qquad (i \neq k)$$

so that $\|e_k - e_i\|$ does not converge to zero as $i, k \to \infty$. However, the following theorem is valid.

THEOREM 1: *If the sequence of vectors $\{f_k\}_1^\infty$ converges weakly to the vector f and if*

$$\lim_{k\to\infty} \|f_k\| = \|f\|,$$

then

$$\lim_{k\to\infty} \| f_k - f \| = 0,$$

i.e., the sequence $\{f_k\}_1^\infty$ converges strongly to the vector f.

Proof: The proof follows from the equation

$$\|f_k - f\|^2 = \|f_k\|^2 - (f_k, f) - (f, f_k) + \|f\|^2.$$

Indeed, by the hypothesis of the theorem,

$$\lim_{k\to\infty} \{ \|f_k\|^2 - (f_k, f) - (f, f_k) + \|f\|^2 \} = 0.$$

An important property of every weakly convergent sequence of vectors is boundedness. The proof of this property does not present any difficulty if the following general proposition is proved first.

THEOREM 2: *If the linear functionals $\Phi_1, \Phi_2, \Phi_3, \ldots$, defined on the space* H, *have the property that the numerical sequence $\{\Phi_k(h)\}_1^\infty$ is bounded for each $h \in$ H, then the sequence $\{\|\Phi_k\|\}_1^\infty$ of the norms of the functionals is bounded.*

Proof: The proof follows almost immediately from the lemma concerning convex functionals proved in Section 18. For $h \in$ H, let

$$p_n(h) = |\Phi_n(h)| \qquad (n = 1, 2, 3, \ldots).$$

The p_n are convex continuous functionals in H. By the lemma just mentioned, the convex functional

$$p(h) = \sup_n p_n(h)$$

is continuous, i.e.,

$$\mathrm{M} = \sup_{\|h\| \leq 1} p(h) < \infty.$$

Consequently,

$$\|\Phi_n\| \leq \mathrm{M} \qquad (n = 1, 2, 3, \ldots)$$

and the theorem is proved.

COROLLARY 1: *Every weakly convergent sequence $\{f_k\}_1^\infty$ is bounded.*

Proof: Each vector f_k determines a functional $\Phi_k(h) = (h, f_k)$. Since the sequence $\{f_k\}_1^\infty$ converges weakly, the numerical sequence $\{\Phi_k(h)\}_1^\infty$ converges for each $h \in$ H and, hence, is bounded. It remains only to apply Theorem 2 and to use the fact that

$$\|\Phi_k\| = \|f_k\|.$$

COROLLARY 2: *Every Hilbert space is weakly complete.*

Proof: Let the sequence of vectors $\{f_k\}_1^\infty$ be fundamental in the sense of weak convergence, i.e., for each $h \in$ H, let

$$\lim_{m, n \to \infty} (f_n - f_m, h) = 0.$$

It follows that the sequence of numbers (f_k, h) $(k=1, 2, 3, \ldots)$ converges for each fixed $h \in$ H. According to Theorem 2 the sequence $\{f_k\}_1^\infty$ is bounded:

$$\|f_k\| \leq \mathrm{M} \qquad (k = 1, 2, 3, \ldots).$$

Therefore, the limit

$$\lim_{k \to \infty} (h, f_k)$$

defines a linear functional $\Phi(h)$ with norm $\leq$ M. According to the representation theorem of F. Riesz, $\Phi(h) = (h, f)$, where f is a unique element of the space H. This element is the weak limit of the sequence $\{f_k\}_1^\infty$.

24. Weak Compactness

A point set is said to be *compact*[11] if every sequence belonging to it contains a convergent subsequence. Corresponding to the two types of convergence (strong and weak) are *strong* (or *ordinary*) *compactness* and *weak compactness*. The concept of compactness is associated with the most important theorem of elementary analysis — the Bolzano-Weierstrass theorem. The conclusion of this theorem is false for a Hilbert space if the theorem is stated in terms of strong convergence. To prove this, it is sufficient to take the infinite orthonormal sequence of vectors $e_1, e_2, e_3, \ldots$. This sequence is bounded, but none of its subsequences is strongly convergent. In connection with what has been said, it may be surprising that the following theorem holds.

THEOREM 1: *Every bounded point set in* H *is weakly compact.*

Proof: Let us take any sequence $\{g_k\}_1^\infty$ of points such that, for some C,

$$\|g_k\| \leqq C < \infty \qquad (k = 1, 2, 3, \ldots).$$

Let L denote the linear envelope of the set $\{g_k\}_1^\infty$, and let $\mathrm{G} = \bar{\mathrm{L}}$ be its closure. Define F by

$$\mathrm{F} = \mathrm{H} \ominus \mathrm{G}.$$

Consider the numerical sequence

$$(1) \qquad (g_1, g_k), \qquad (k = 1, 2, 3, \ldots).$$

It is bounded because

$$|(g_1, g_k)| \leqq \|g_1\| \cdot \|g_k\| \leqq C^2 \qquad (k = 1, 2, 3, \ldots).$$

Therefore, the sequence (1) contains a convergent subsequence. In other words, $\{g_k\}_1^\infty$ contains a subsequence $\{g_{1k}\}_{k=1}^\infty$ for which

$$\lim_{k\to\infty} (g_1, g_{1k})$$

exists. Similarily, from the boundedness of the numerical sequence

$$(2) \qquad (g_2, g_{1k})$$

we conclude that $\{g_{1k}\}_{k=1}^\infty$ contains a subsequence $\{g_{2k}\}_{k=1}^\infty$ for which

$$\lim_{k\to\infty} (g_2, g_{2k})$$

exists. Repeating this argument, we get an infinite sequence of sequences

$$\begin{array}{l} g_{11}, g_{12}, g_{13}, \ldots, \\ g_{21}, g_{22}, g_{23}, \ldots, \\ g_{31}, g_{32}, g_{33}, \ldots, \\ \ldots \; \ldots \; \ldots \; \ldots \end{array}$$

each of which is a subsequence of the preceding. It is evident that the diagonal sequence

[11] Translator's Note: This general concept is often called *sequential compactness.*

$$g_{11}, g_{22}, g_{33}, \ldots$$

has the property that, for each integer r

$$\lim_{k\to\infty} (g_r, g_{kk})$$

exists. Hence, it follows that

$$\lim_{k\to\infty} (g, g_{kk})$$

exists for each $g \in \mathrm{L}$ and, therefore, for each $g \in \mathrm{G}$. If $f \in \mathrm{F}$, then

$$(f, g_{kk}) = 0, \qquad (k = 1, 2, 3, \ldots).$$

Consequently,

$$\lim_{k\to\infty} (f, g_{kk})$$

exists for each $f \in \mathrm{F}$. Since $\mathrm{H} = \mathrm{F} \oplus \mathrm{G}$, the results we have obtained imply the existence, for each $h \in \mathrm{H}$, of

$$\lim_{k\to\infty} (h, g_{kk}).$$

Therefore, the sequence $\{g_{kk}\}_{k=1}^{\infty}$ is fundamental in the sense of weak convergence. By the weak completeness of the space, this sequence converges weakly to some element of H, and this proves our theorem.

THEOREM 2: *For the weak convergence of the sequence of vectors* $\{g_k\}_1^{\infty}$ *it is necessary and sufficient that:*

1. *the numerical sequence*

$$(g_k, f) \qquad (k = 1, 2, 3, \ldots)$$

converge for each f of some set M *which is dense in* H; *and*

2. *the sequence* $\{g_k\}_{k=1}^{\infty}$ *be bounded, i.e, the inequality*

$$\|g_k\| \leqq C < \infty \qquad (k = 1, 2, 3, \ldots)$$

hold for some C.

Proof: The necessity of condition 1 is evident. The necessity of condition 2 is indicated by Corollary 1 of Theorem 2 of Section 23. We turn to the proof of the sufficiency of the conditions mentioned. By Theorem 1 of the preceding paragraph, $\{g_k\}_{k=1}^{\infty}$ has a weakly convergent subsequence $\{g_{k_i}\}_{i=1}^{\infty}$. Let g be the weak limit of this subsequence.
Then

$$\lim_{i\to\infty} (h, g_{ki}) = (h, g).$$

According to condition 1 of the theorem,

$$\lim_{k\to\infty} (f, g_k)$$

exists for each $f \in \mathrm{M}$. Therefore,

$$\lim_{k\to\infty} (f, g_k) = (f, g)$$

for $f \in \mathrm{M}$, and it remains to prove (we leave this to the reader) that this equation holds if f is any element of H.

25. A Criterion for the Boundedness of an Operator

THEOREM: *Let A and A^* be linear operators defined on* H *and assume that*

$$(Af, g) = (f, A^*g)$$

for $f, g \in$ H. Then A is bounded, and A^* is the adjoint of A.

Proof: We assume the contrary and suppose that there exists a sequence of vectors $\{f_k\}_1^\infty$ such that

$$\|f_k\| = 1, \quad \|Af_k\| > k \qquad (k = 1, 2, 3, \ldots).$$

The expressions

$$(g, Af_k) = \Phi_k(g) \qquad (k = 1, 2, 3, \ldots)$$

define linear functionals Φ_k in H. Since

$$\Phi_k(g) = (A^*g, f_k) \qquad (k = 1, 2, 3, \ldots),$$

the numerical sequence $\{\Phi_k(g)\}_{k=1}^\infty$ is bounded. By Theorem 2 of Section 23, the sequence of norms $\|\Phi_k\|$ $(k = 1, 2, 3, \ldots)$, i.e., the sequence of numbers $\|Af_k\|$, is also bounded, which is a contradiction. Thus, the theorem is proved.

An important special case of this theorem is due to Hellinger and Toeplitz. We mention it in the following section.

26. Linear Operators in a Separable Space

In this section we shall consider linear operators defined everywhere on a separable Hilbert space H. We show that bounded operators admit matrix representations completely analogous to the well known matrix representations of operators on finite dimensional spaces.

We choose any orthonormal basis $\{e_k\}_1^\infty$ in H and let[12]

$$Ae_k = c_k, \qquad (k = 1, 2, 3, \ldots)$$

and

$$(Ae_k, e_i) = a_{ik} \qquad (i, k = 1, 2, 3, \ldots). \tag{1}$$

Thus

$$c_k = \sum_{i=1}^\infty a_{ik} e_i \qquad (k = 1, 2, 3, \ldots).$$

Moreover,

$$\sum_{i=1}^\infty |a_{ik}|^2 < \infty \qquad (k = 1, 2, 3, \ldots).$$

[12] We remark that if the operator A is not defined everywhere in H, but only on a set D which is dense in H, then there exists in H an orthonormal basis $\{e_k\}_1^\infty$, the elements of which belong to D.

We introduce the infinite matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

of which the elements of the kth column are the components of the vector into which the operator A maps the kth coordinate vector. If the operator A is bounded, then it is uniquely determined by the infinite matrix (a_{ik}). For the proof of this assertion it is necessary to show how to represent the operator in terms of the matrix and the orthonormal basis $\{e_k\}_1^\infty$. First, we have

$$Ae_k = \sum_{i=1}^{\infty} a_{ik} e_i \qquad (k = 1, 2, 3, \dots).$$

Since the operator A is linear, it is well defined on the linear envelope of the given basis, i.e., for all vectors each of which has only a finite number of nonzero components relative to the basis. Since A is continuous, the value of Af for an arbitrary vector $f \in \mathrm{H}$ may be found by means of a passage to a limit.

It is not difficult to write a simple formula for the components of the vector f; indeed, if

(2) $$f = \sum_{k=1}^{\infty} x_k e_k$$

then

(3) $$Af = \sum_{k=1}^{\infty} y_k e_k,$$

where

(4) $$y_k = \sum_{i=1}^{\infty} a_{ki} x_i.$$

In fact, if

$$f_n = \sum_{k=1}^{n} x_k e_k,$$

then

$$Af_n = \sum_{k=1}^{\infty} y_k^{(n)} e_k$$

where

$$y_k^{(n)} = \sum_{i=1}^{n} a_{ki} x_i.$$

By the boundedness of the operator A,

$$y_k = (Af, e_k) = \lim_{n\to\infty} (Af_n, e_k) = \lim_{n\to\infty} y_k^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{n} a_{ki} x_i = \sum_{i=1}^{\infty} a_{ki} x_i.$$

DEFINITION: *If the operator A is defined everywhere in* H *and if its value for any vector* (2) *is given by the formulas* (3) *and* (4), *then we say that the operator A admits a matrix representation relative to the orthogonal basis* $\{e_k\}_1^\infty$.

Thus, we have proved that every bounded linear operator defined on the entire space admits a matrix representation with respect to each orthogonal basis. This is the analogue, mentioned at the very beginning of the present section, between a separable Hilbert space and a finite-dimensional space, with respect to bounded linear operators.

THEOREM: *If an operator A, defined everywhere in a separable space* H, *admits a matrix representation with respect to some orthogonal basis, then it is bounded.*

(This proposition is a frequently used special case of the theorem of the preceding section, mentioned above, which is due to Hellinger and Toeplitz).[13]

Proof: By hypothesis, the series

$$(Af, e_k) = \sum_{i=1}^{\infty} a_{ki} x_i \qquad (k = 1, 2, 3, \ldots)$$

converges for each vector

$$f = \sum_{k=1}^{\infty} x_k e_k,$$

where $\{e_k\}_1^\infty$ is the orthonormal basis, mentioned in the theorem, with respect to which the operator A admits a matrix representation. Therefore, by the theorem of Landau (see Section 18),

$$(5) \qquad \sum_{i=1}^{\infty} |a_{ki}|^2 < \infty \qquad (k = 1, 2, 3, \ldots).$$

We introduce the sequence of vectors

$$c_k^* = \sum_{i=1}^{\infty} \bar{a}_{ki} e_i \qquad (k = 1, 2, 3, \ldots)$$

and by means of them, define the linear operator A^*. First, let

$$A^* e_k = c_k^* \qquad (k = 1, 2, 3, \ldots)$$

and then use linearity to define A^* on the linear envelope of the set of vectors e_k. Finally, extend A^* by continuity to all of H. It is easy to prove that for any $f, g \in$ H,

$$(Af, g) = (f, A^* g)$$

after which, to complete the proof, it remains to apply the theorem of the preceding section.

[13] E. Hellinger and O. Toeplitz [1].

We shall not present all the details of the proof just outlined, but we mention another proof of the theorem, which is based directly on the lemma concerning convex functionals in Section 18. In view of inequality (5), the expression

$$\Phi_k(f) = \sum_{i=1}^{\infty} a_{ki} x_i \qquad (k = 1, 2, 3, \ldots)$$

defines a linear functional of

$$f = \sum_{k=1}^{\infty} x_k e_k.$$

Therefore,

$$p_n(f) = \sqrt{\sum_{k=1}^{n} |\Phi_k(f)|^2} \qquad (n = 1, 2, 3, \ldots)$$

defines a convex continuous functional of f. Since

$$\sum_{k=1}^{n} |\Phi_k(f)|^2 = \sum_{k=1}^{n} |(Af, e_k)|^2 \leqq \|Af\|^2$$

the sequence $\{p_n(f)\}_{n=1}^{\infty}$ is bounded for each $f \in \mathrm{H}$. On the basis of the corollary of the lemma concerning convex functionals, the functional

$$p(f) = \sup_n p_n(f) = \lim_{n\to\infty} p_n(f) = \sqrt{\sum_{k=1}^{\infty} |\Phi_k(f)|^2} = \|Af\|$$

is continuous, i.e., there exists a constant M such that

$$p(f) \leqq M \|f\|.$$

But this implies that the operator A is bounded.

The proof of the theorem can be formulated also in the following form: if for arbitrary numbers $x_k (k=1, 2, 3, \ldots)$ such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty$$

the inequality

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 < \infty$$

holds, then there exists a constant M such that

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 \leqq M^2 \sum_{k=1}^{\infty} |x_k|^2.$$

This reduces to the theorem of E. Landau (see Section 18) if $a_{ki} = 0$ for $k > 1$.

Let us agree to write

$$A \sim (a_{ik})$$

if the bounded linear operator A, defined everywhere in H, corresponds to the matrix (a_{ik}) according to (1). Here the orthogonal basis $\{e_k\}_1^\infty$ is arbitrary but fixed.

If

$$A \sim (a_{ik}), \qquad B \sim (b_{ik})$$

then, as is easily verified,

$$AB \sim (c_{ik})$$

where

$$c_{ik} = \sum_{r=1}^{\infty} a_{ir} b_{rk} \qquad (i, k = 1, 2, 3, \ldots).$$

If we define matrix multiplication by means of this equation, then

$$AB \sim (a_{ir}) \cdot (b_{rk}).$$

Furthermore, if

$$A \sim (a_{ik})$$

and

$$A^* \sim (a_{ik}^*)$$

then

$$a_{ik}^* = \bar{a}_{ki} \qquad (i, k = 1, 2, 3, \ldots).$$

Therefore, the condition that the bounded operator A be self-adjoint may be expressed in the form

$$a_{ik} = \bar{a}_{ki}. \tag{6}$$

Matrices for which equation (6) holds are called *symmetric* or *Hermitean.*

A bilinear functional is generated by the operator A by means of the equation

$$(Af, g) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ki} x_i \right) \bar{y}_k.$$

In this equation

$$f = \sum_{k=1}^{\infty} x_k e_k, \qquad g = \sum_{k=1}^{\infty} y_k e_k.$$

In the double sum appearing above it is possible to reverse the order of summation, since the equation

$$(Af, g) = (f, A^* g)$$

implies that

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ki} x_i \right) \bar{y}_k = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{ki} \bar{y}_k \right) x_i.$$

From the inequality

$$|(Af, g)| \leq M \|f\| \cdot \|g\| \tag{7}$$

it follows that

$$\left| \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ki} x_i \bar{y}_k \right| \leq M \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{k=1}^{\infty} |y_k|^2}.$$

If each of the vectors f and g has only a finite number of nonzero components, then the last inequality may be written in the form

$$(8) \qquad \left| \sum_{i=1}^{p} \sum_{k=1}^{q} a_{ki} x_i \bar{y}_k \right| \leq M \sqrt{\sum_{i=1}^{p} |x_i|^2} \sqrt{\sum_{k=1}^{q} |y_k|^2}.$$

THEOREM: *In order that the matrix* (a_{ik}) *represent a bounded linear operator defined everywhere in* H, *it is necessary and sufficient that, for some constant* M, *the inequality* (8) *hold for any numbers* $x_1, x_2, \ldots, x_p$ *and* $y_1, y_2, \ldots, y_q$.

Proof: If the operator A is bounded and

$$a_{ik} = (Ae_k, e_i) \qquad (i, k = 1, 2, 3, \ldots),$$

then (7) implies (8). Now let (a_{ik}) be a matrix which satisfies condition (8). We shall show that this matrix determines a bounded linear operator A. First, from (8) with

$$x_1 = x_2 = \ldots = x_{k-1} = x_{k+1} = \ldots = 0, x_k \neq 0,$$
$$y_1 = y_2 = \ldots = y_{m-1} = y_{n+1} = y_{n+2} = \ldots = 0$$

we get

$$\left| \sum_{i=m}^{n} a_{ik} \bar{y}_i \right| \leq M \sqrt{\sum_{i=m}^{n} |y_i|^2}.$$

This implies the convergence of the series

$$\sum_{i=1}^{\infty} a_{ik} \bar{y}_i$$

for any sequence $\{y_i\}_1^\infty$ in l^2. Hence, the theorem of Landau (see Section 18) implies the convergence of the series

$$\sum_{i=1}^{\infty} | a_{ik} |^2 \qquad (k = 1, 2, 3, \ldots).$$

We define the operator A_0, first for the basis elements by the formula

$$A_0 e_k = \sum_{i=1}^{\infty} a_{ik} e_i \qquad (k = 1, 2, 3, \ldots),$$

and then, by means of linearity, for all vectors with only a finite number of components different from zero. Now we prove that the operator A_0 is bounded. We have, by (8) for f and g with only finite number of nonzero components,

$$(9) \qquad |(A_0 f, g)| \leq M \|f\| \cdot \|g\|.$$

By the continuity of the scalar product, the inequality (9) is satisfied for all $g \in$ H. Let

$$g = A_0 f$$

in (9) to get

$$\|A_0 f\|^2 \leq M \|f\| \cdot \|A_0 f\|$$

so that

$$\|A_0 f\| \leqq M\|f\|.$$

Thus, A_0 is bounded. Extending A_0 by continuity to the whole space H, we get the bounded operator A and the correspondence

$$A \sim (a_{ik}).$$

The theorem is proved.

We note that if the matrix (a_{ik}) is symmetric (Hermitean), i.e., if

$$a_{ik} = \bar{a}_{ki}$$

then it is possible to replace the condition (8) by (see Section 22)

$$\left| \sum_{i,k=1}^{p} a_{ik} x_i \bar{x}_k \right| \leqq M \sum_{i=1}^{p} |x_i|^2.$$

We now give an example of the matrix representation of a bounded linear operator. Consider the *integral operator* in $L^2(-\infty, \infty)$ defined by the formula

$$g(s) = Af(s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt$$

where the function $K(s, t)$ is called the *kernel* of the operator. If the kernel satisfies the condition

$$\text{(10)} \qquad \iint_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt < \infty,$$

it is called a *Hilbert-Schmidt kernel*, and the operator determined by it is called a *Hilbert-Schmidt operator*. We suppose that the condition (10) is satisfied. Then, for almost all t and u

$$\int_{-\infty}^{\infty} |K(s,t) \cdot \overline{K(s,u)}|\, ds \leqq \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds} \sqrt{\int_{-\infty}^{\infty} |K(s,u)|^2 ds}\,.$$

But since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t)| \cdot |f(u)| \cdot \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds} \sqrt{\int_{-\infty}^{\infty} |K(s,u)|^2 ds\, dt\, du} =$$

$$= \left\{ \int_{-\infty}^{\infty} |f(t)| \sqrt{\int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt} \right\}^2 \leqq$$

$$\leqq \int_{-\infty}^{\infty} |f(t)|^2 dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt < \infty,$$

we have, by Fubini's theorem,

$$\| g \| = \left\{ \int_{-\infty}^{\infty} ds \left| \int_{-\infty}^{\infty} K(s,t) f(t)\, dt \right|^2 \right\}^{\frac{1}{2}} =$$

$$= \left\{ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} K(s,t) f(t)\, dt \int_{-\infty}^{\infty} \overline{K(s,u)}\, \overline{f(u)}\, du \right\}^{\frac{1}{2}} \leqq$$

$$\leqq \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt} \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt}\ .$$

We see that the Hilbert-Schmidt operator is bounded and that its norm does not exceed the quantity

$$\sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(s,t)|^2 ds\, dt.}$$

Let us take in $L^2(-\infty, \infty)$ any complete orthonormal system of functions $\{\varphi_k(t)\}_1^\infty$ and define

$$a_{ik} = \iint_{-\infty}^{\infty} K(s,t)\, \overline{\varphi_i(s)}\, \varphi_k(t)\, ds\, dt \qquad (i, k = 1, 2, 3, \ldots).$$

Choose any $f(t) \in L^2(-\infty, \infty)$ and let

$$\int_{-\infty}^{\infty} f(t)\, \overline{\varphi_k(t)}\, dt = x_k \qquad (k = 1, 2, 3, \ldots).$$

Then the Fourier coefficients

$$y_i = \int_{-\infty}^{\infty} g(s)\, \overline{\varphi_i(s)}\, ds \qquad (i = 1, 2, 3, \ldots)$$

of the function

$$g(s) = \int_{-\infty}^{\infty} K(s,t) f(t)\, dt$$

are given by

$$y_i = \iint_{-\infty}^{\infty} K(s,t) f(t)\, \overline{\varphi_i(s)}\, ds\, dt =$$

$$= \int_{-\infty}^{\infty} f(t) \left\{ \int_{-\infty}^{\infty} K(s,t)\, \overline{\varphi_i(s)}\, ds \right\} dt \qquad (i = 1, 2, 3, \ldots).$$

But since

$$(11)\qquad \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \sim \sum_{k=1}^{\infty} a_{ik}\,\overline{\varphi_k(t)} \qquad (i=1,2,3,\ldots)$$

and

$$f(t) \sim \sum_{k=1}^{\infty} x_k\,\varphi_k(t),$$

we have, by the Parseval relation,

$$y_i = \sum_{k=1}^{\infty} a_{ik}\,x_k \qquad (i=1,2,3,\ldots).$$

In a similar way, it follows from relation (11) that

$$\int_{-\infty}^{\infty} dt \left| \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \right|^2 = \sum_{k=1}^{\infty} |a_{ik}|^2 \qquad (i=1,2,3,\ldots).$$

But on the other hand, by the Parseval relation,

$$\int_{-\infty}^{\infty} |K(s,t)|^2\,ds = \sum_{i=1}^{\infty} \left| \int_{-\infty}^{\infty} K(s,t)\,\overline{\varphi_i(s)}\,ds \right|^2$$

and, therefore,

$$\iint_{-\infty}^{\infty} |K(s,t)|^2\,ds\,dt = \sum_{i,k=1}^{\infty} |a_{ik}|^2.$$

We see that each Hilbert-Schmidt operator is represented by a matrix operator for which

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty.$$

27. Completely Continuous Operators

Hilbert considered first the important class of completely continuous operators. A linear operator A defined everywhere in H is said to be *completely continuous* if it maps each bounded point set into a set which is compact in the sense of strong convergence.

A completely continuous operator A is bounded. In fact, otherwise there would exist a sequence of points f_k $(k=1, 2, 3, \ldots)$ for which

$$\|f_k\| = 1, \ \|Af_k\| > k \qquad (k = 1, 2, 3, \ldots)$$

but this is impossible, since the point set $\{Af_k\}_1^\infty$ is compact.

Completely continuous operators have another definition: a linear operator A defined everywhere in H is completely continuous if it maps every weakly convergent sequence into a strongly convergent sequence. The proof of the equivalence of these definitions we leave to the reader.

We also leave to the reader the proofs of the following simple facts:

1. If A is a completely continuous operator and if B is a bounded operator defined everywhere in H, then the operators AB and BA are completely continuous.

2. If A_1 and A_2 are completely continuous operators, then $\alpha_1 A_1 + \alpha_2 A_2$ is a completely continuous operator.

THEOREM: *If A is a bounded linear operator defined everywhere in* H, *and if the operator A^*A is completely continuous, then the operator A is completely continuous.*

Proof: Let M be any bounded infinite set of points f ($\|f\| \leqq C$). Let $\{f_k\}_1^\infty$ be any sequence of elements of this set. This sequence is mapped by the operator A^*A into a strongly convergent sequence. Since

$$\|Af_n - Af_m\|^2 = (A(f_n - f_m), A(f_n - f_m)) =$$
$$= (A^*A(f_n - f_m), f_n - f_m) \leqq \|A^*Af_n - A^*Af_m\| \cdot \|f_n - f_m\|,$$

we have

$$\lim_{m,n\to\infty} \|A^*Af_n - A^*Af_m\| = 0$$

where

$$\|f_n - f_m\| \leqq 2C.$$

Therefore,

$$\lim_{m,n\to\infty} \|Af_n - Af_m\| = 0,$$

so that the sequence $\{Af_n\}_1^\infty$ converges and the theorem is proved.

COROLLARY: *If the operator A is completely continuous, then the operator A^* has the same property.*

Proof: In fact, if the operator A is completely continuous, then the operator $AA^* = (A^*)^*A^*$ is completely continuous, and it remains only to apply the theorem just proved.

28. A Criterion for Complete Continuity of an Operator

The following theorem is often used to prove that a given operator is completely continuous.

THEOREM: *If for each $\varepsilon > 0$ there exists a completely continuous operator A_ε such that*

$$\|(A - A_\varepsilon)f\| \leqq \varepsilon\|f\|$$

for $f \in$ H then the operator A is completely continuous.

Proof: We choose a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \ldots$ ($\lim_{n\to\infty} \varepsilon_n = 0$) and consider a sequence of completely continuous operators $A_{\varepsilon_1}, A_{\varepsilon_2}, \ldots$ corresponding to it by the condition of the theorem. Let M be an arbitrary bounded set of points f($\|f\| \leqq C$) in the space H. Let us take an arbitrary sequence $\{f_k\}_1^\infty$ of points belonging to M. According to the complete continuity of A_{ε_1} there exists a subsequence

$$f_{11}, f_{12}, f_{13}, \ldots \tag{1}$$

which is mapped by the operator A_{ε_1} into a convergent sequence. From the sequence (1) we select a subsequence

$$f_{21}, f_{22}, f_{23}, \ldots \tag{2}$$

which is mapped into a convergent sequence by the operator A_{ε_2}. Continuing this process, we get the infinite sequence of sequences

$$\begin{array}{l} f_{11}, f_{12}, f_{13}, \ldots, \\ f_{21}, f_{22}, f_{23}, \ldots, \\ f_{31}, f_{32}, f_{33}, \ldots, \\ \ldots\ldots\ldots\ldots, \end{array}$$

such that each is a subsequence of the preceding. The diagonal sequence

$$f_{11}, f_{22}, f_{33}, \ldots$$

is mapped into a strongly convergent sequence by each of the operators A_{ε_i}. We prove next that the diagonal sequence $\{f_{kk}\}_{k=1}^\infty$ is mapped into a convergent sequence also by the operator A. For this it suffices to prove that

$$\lim_{m,\, n\to\infty} \|Af_{nn} - Af_{mm}\| = 0. \tag{3}$$

We have the inequality

$$\|Af_{nn} - Af_{mm}\| \leqq \|(A - A_{\varepsilon_k})f_{nn}\| + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\| + \\ + \|(A - A_{\varepsilon_k})f_{mm}\| \leqq 2\varepsilon_k C + \|A_{\varepsilon_k}f_{nn} - A_{\varepsilon_k}f_{mm}\|.$$

By taking k sufficiently large, we can make the term $2\varepsilon_k C$ as small as desired. After this, we can take N so large that the second term of the last member is made as small as desired for $m, n > N$. Thus, the relation (3) is proved.

We make use of the theorem just proved in order to establish the complete continuity of the matrix operator defined by

$$y_i = \sum_{k=1}^\infty a_{ik}x_k \qquad (i = 1, 2, 3 \ldots),$$

where

$$\sum_{i,\, k=1}^\infty |a_{ik}|^2 < \infty, \tag{4}$$

which implies the complete continuity of every integral operator with a Hilbert-Schmidt kernel. From (4) it follows that

$$\lim_{p\to\infty} \sum_{i=1}^p \sum_{k=1}^\infty |a_{ik}|^2 < \infty.$$

Therefore, for each $\varepsilon > 0$ there exists an integer $p = p(\varepsilon)$ such that

$$\sum_{i=p+1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 \leqq \varepsilon^2.$$

Now we construct the operator A_ε with the aid of the relation

$$A_\varepsilon f = y_1 e_1 + y_2 e_2 + \ldots + y_p e_p$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k \qquad (i = 1, 2, 3, \ldots)$$

if

$$f = \sum_{k=1}^{\infty} x_k e_k.$$

Let

$$Af = \sum_{i=1}^{\infty} y_i e_i.$$

We have

$$\| Af - A_\varepsilon f \|^2 = \sum_{i=p+1}^{\infty} |y_i|^2 =$$

$$= \sum_{i=p+1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} x_k \right|^2 \leqq \sum_{i=p+1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 \|f\|^2 \leqq \varepsilon^2 \|f\|^2.$$

It remains only to verify that the operator A_ε is completely continuous. Choose any bounded set of vectors in H. The operator A_ε maps this set into a bounded set in a finite-dimensional subspace of H, and this set is compact by the classical Bolzano-Weierstrass theorem.

We emphasize the fact that the convergence of the series

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2$$

is only a sufficient, but not a necessary condition for the complete continuity of the matrix operator. In the special case when the numbers a_{ik} satisfy the relation

$$a_{ik} = 0 \text{ for } |i - k| > r \qquad (i, k = 1, 2, 3, \ldots)$$

for some fixed r, it is possible to specify a necessary and sufficient condition for complete continuity. It is expressed by the relation

$$\lim_{i,k \to \infty} a_{ik} = 0.$$

For simplicity we sketch the proof only in the case with $r = 1$. In this case, the matrix defining the operator has the form

$$\left(\begin{array}{cccccc} \alpha_1 & \beta_1 & 0 & 0 & 0 & \dots \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & 0 & \dots \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 & \dots \\ 0 & 0 & \gamma_3 & \alpha_4 & \beta_4 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \tag{5}$$

and is called a *Jacobi matrix*. For $r > 1$ the matrix is called a *generalized Jacobi matrix*. Let the operator A, determined by the matrix (5), be completely continuous. Then the sequence of vectors

$$Ae_i = \beta_{i-1} e_{i-1} + \alpha_i e_i + \gamma_i e_{i+1}$$
$$(\beta_0 = 0, \ i = 1, 2, 3, \dots)$$

must converge strongly. Supposing that the matrix A does not satisfy the condition in question, we select a sequence $i_1, i_2, i_3, \dots$ such that

$$i_k \geqq i_{k-1} + 3$$

and

$$|\beta_{i_k - 1}|^2 + |\alpha_{i_k}|^2 + |\gamma_{i_k}|^2 \to \delta > 0$$

where $\delta \leqq \infty$. A simple computation yields

$$\| Ae_{i_n} - Ae_{i_m} \|^2 = |\beta_{i_n - 1}|^2 + |\alpha_{i_n}|^2 + |\gamma_{i_n}|^2 +$$
$$+ |\beta_{i_m - 1}|^2 + |\alpha_{i_m}|^2 + |\gamma_{i_m}|^2 \to 2\delta \neq 0.$$

This contradicts the strong convergence of the sequence $\{Ae_k\}_1^\infty$.

We now prove the sufficiency of the assertion. Let

$$\alpha_k \to 0, \ \beta_k \to 0, \ \gamma_k \to 0 \qquad (k \to \infty)$$

and let the sequence $\{f^{(k)}\}_1^\infty$ converge weakly to f. Because

$$Af^{(n)} = \sum_{k=1}^{\infty} x_k^{(n)} Ae_k = \sum_{k=1}^{\infty} x_k^{(n)} (\beta_{k-1} e_{k-1} + \alpha_k e_k + \gamma_k e_{k+1}) =$$
$$= \sum_{k=1}^{\infty} (\beta_k x_{k+1}^{(n)} + \alpha_k x_k^{(n)} + \gamma_{k-1} x_{k-1}^{(n)}) e_k \qquad (\gamma_0 = 0)$$

we have

$$\| Af^{(n)} - Af^{(m)} \|^2 =$$

$$= \sum_{k=1}^{\infty} | \beta_k \{x_{k+1}^{(n)} - x_{k+1}^{(m)}\} + \alpha_k \{x_k^{(n)} - x_k^{(m)}\} + \gamma_{k-1} \{x_{k-1}^{(n)} - x_{k-1}^{(m)}\} |^2$$

$$= \sum_{k=1}^{q} + \sum_{k=q+1}^{\infty} .$$

The first term on the right side tends toward zero for fixed q as $m, n \to \infty$. Therefore, it is sufficient to show that it is possible to make the second term on the right side as small as desired for all m and n by taking q sufficiently large. But, if q is sufficiently large and $k > q$, then

$$|\beta_k| < \varepsilon, |\alpha_k| < \varepsilon, |\gamma_{k-1}| < \varepsilon.$$

Therefore,

$$\sum_{k=q+1}^{\infty} |\beta_k\{x_{k+1}^{(n)} - x_{k+1}^{(m)}\} + \alpha_k\{x_k^{(n)} - x_k^{(m)}\} + \gamma_{k-1}\{x_{k-1}^{(n)} - x_{k-1}^{(m)}\}|^2 \leqq$$
$$\leqq 9\varepsilon^2 \|f^{(n)} - f^{(m)}\|^2 .$$

Thus, our assertion is proved.

29. Sequences of Bounded Linear Operators

We distinguish three modes of convergence of a sequence $\{A_n\}_1^\infty$ of bounded linear operators defined everywhere in H: *weak* convergence, *strong* convergence (or, simply, convergence), and *uniform* convergence. A sequence $\{A_n\}_1^\infty$

converges weakly to the operator A $A_n \xrightarrow{w} A$	if for each $f \in$ H $A_n f \xrightarrow{w} Af \quad (n\to\infty)$
converges strongly to the operator A $A_n \to A$	if for each $f \in$ H $A_n f \to Af \quad (n\to\infty)$
converges uniformly to the operator A $A_n \rightrightarrows A$	if $\|A_n - A\| \to 0 \quad (n\to\infty)$

If a sequence of operators converges uniformly, then it also converges strongly; if it converges strongly, then it also converges weakly.

Using the results of Section 23 and the lemma about convex functionals in Section 18, it is possible to prove the following proposition: if the sequence $\{A_n\}_1^\infty$ of bounded linear operators defined everywhere in H converges weakly, then the sequence $\{\|A_n\|\}_1^\infty$ of the norms of these operators is bounded.

We mention one more proposition, which is analogous to Corollary 2 in Section 23: if the sequence of bilinear functionals $\{\Omega_n(f, g)\}_1^\infty$ has the property that for arbitrary f and g the limit

$$\lim_{n\to\infty} \Omega_n(f, g) = \omega(f, g)$$

exists and is finite, then this limit defines a bilinear functional. As is easily seen, it is sufficient to prove that

$$|\omega(f,g)| \leqq C < \infty$$

for $\|f\| \leqq 1$, $\|g\| \leqq 1$, and some number C. Each of the bilinear functionals $\Omega_n(f, g)$ is determined by a particular bounded linear operator:

$$\Omega_n(f, g) = (A_n f, g).$$

The hypothesis implies that the sequence of operators $\{A_n\}_1^\infty$ converges weakly. Consequently, for $f, g \in$ H,

$$|(A_n f, g)| \leqq C \|f\| \cdot \|g\|$$

or equivalently,

$$|\Omega_n(f,g)| \leqq C \|f\| \cdot \|g\|.$$

Hence, it follows that

$$|\omega(f,g)| \leqq C \|f\| \cdot \|g\|.$$

Chapter III

PROJECTION OPERATORS AND UNITARY OPERATORS

30. Definition of a Projection Operator

Let G be a subspace of the space H and let

$$F = H \ominus G,$$

so that

$$H = G \oplus F.$$

Then each vector $h \in H$ is uniquely representable in the form

$$h = g + f,$$

where $g \in G$ and $f \in F$. In Section 7 the vector g was called the projection of h on G. The operator which maps each $h \in H$ into its projection g on G is called the operator of projection on G or, simply, a projection operator. It is denoted by P_G or sometimes, when the subspace G is specified in advance, by P. Thus, if g and h are related as above,

$$g = Ph = P_G h.$$

A projection operator is evidently linear. In addition, it is bounded and its norm is equal to one. Indeed, since the equation

$$\|h\|^2 = \|g\|^2 + \|f\|^2$$

implies that

(1) $$\|g\| \leqq \|h\|, \Rightarrow \|Ph\| \leq \|h\|$$

we have

$$\|P\| \leqq 1.$$

But if $h \in G$, then $g = h$, so that there can be equality in (1). Therefore,

$$\|P\| = 1.$$

31. Properties of Projection Operators

From the definition of a projection operator it follows easily that

1) $P^2 = P$,

2) $P^* = P$.

Indeed, if $P = P_G$ then for an arbitrary $h \in H$ the vector $g = Ph$ belongs to G, so that $Pg = g$ and $P^2h = Ph$, and this implies that $P^2 = P$.

In order to prove that P is self-adjoint, we choose two arbitrary vectors $h_1, h_2 \in \mathrm{H}$ and let

$$h_1 = g_1 + f_1, h_2 = g_2 + f_2,$$

where $g_1 = Ph_1$ and $g_2 = Ph_2$. Then

$$(g_1, h_2) = (g_1, g_2) = (h_1, g_2),$$

so that

$$(Ph_1, h_2) = (h_1, Ph_2)$$

for $h_1, h_2 \in \mathrm{H}$. This implies that $P^* = P$.

From the properties just established it follows that

$$(Ph, h) \geqq 0.$$

In fact,

$$(Ph, h) = (P^2h, h) = (Ph, P^*h) = (Ph, Ph) \geqq 0.$$

Now we prove that the properties 1), 2) characterize a projection operator.

THEOREM: *If P is any operator defined on* H *such that, for arbitrary* $h_1, h_2 \in \mathrm{H}$,

(1) $$(P^2h_1, h_2) = (Ph_1, h_2),$$

(2) $$(Ph_1, h_2) = (h_1, Ph_2)$$

then there exists a subspace $\mathrm{G} \subset \mathrm{H}$ *such that P is the operator of projection on* G.

Proof: The operator P is bounded. This follows from (2) and a theorem in Section 25. However, it can be proved also by the following simple argument. We have

$$\| Ph \|^2 = (Ph, Ph) = (P^2h, h) = (Ph, h)$$

and

$$\| Ph \|^2 \leqq \| Ph \| \cdot \| h \|,$$

so that

$$\| Ph \| \leqq \| h \|.$$

Thus, the operator P is bounded and its norm is not greater than 1. We denote by G the set of all vectors $g \in \mathrm{H}$ for which

$$Pg = g.$$

Clearly, G is a linear manifold. We shall prove that G is closed so that it is also a subspace. Let $g_n \in \mathrm{G}$ ($n = 1, 2, 3, \ldots$) and $g_n \to g$. Then

$$g_n = Pg_n$$

and

$$Pg - g_n = Pg - Pg_n = P(g - g_n),$$

so that

$$\| Pg - g_n \| \leqq \| g - g_n \|.$$

Let $n \to \infty$ to get

$$\|Pg - g\| \leqq 0,$$

so that

$$Pg = g.$$

Hence, $g \in \mathrm{G}$, which implies that G is closed. We must prove that $P = P_{\mathrm{G}}$, where P_{G} is the operator of projection on G. For each $h \in \mathrm{H}$, the vector Ph belongs to G because $P(Ph) = Ph$. The subspace G also contains $P_{\mathrm{G}}h$. Therefore, it is sufficient to prove that

$$(Ph - P_{\mathrm{G}}h, g') = 0$$

or

$$(Ph, g') = (P_{\mathrm{G}}h, g')$$

for each $g' \in \mathrm{G}$. But this follows from the equations

$$(Ph, g') = (h, Pg') = (h, g'),$$
$$(P_{\mathrm{G}}h, g') = (h, P_{\mathrm{G}}g') = (h, g').$$

To conclude the present section, we remark that if G is a subspace and E is the identity operator, then $E - P$ is the operator of projection on $\mathrm{H} \ominus \mathrm{G}$.

32. Operations Involving Projection Operators

In the present section we shall prove a few simple propositions concerning the multiplication, addition and subtraction of projection operators.

THEOREM 1: *The product of two projection operators P_{G_1} and P_{G_2} is also a projection operator if and only if P_{G_1} and P_{G_2} commute, i.e., if*

$$P_{\mathrm{G}_1}P_{\mathrm{G}_2} = P_{\mathrm{G}_2}P_{\mathrm{G}_1}.$$

In this case,

$$P_{\mathrm{G}_1}P_{\mathrm{G}_2} = P_{\mathrm{G}},$$

where $\mathrm{G} = G_1 \cap G_2$.[1]

Proof: First, let the product be a projection operator. Then

$$P_{\mathrm{G}_1}P_{\mathrm{G}_2} = (P_{\mathrm{G}_1}P_{\mathrm{G}_2})^* = P^*_{\mathrm{G}_2}P^*_{\mathrm{G}_1} = P_{\mathrm{G}_2}P_{\mathrm{G}_1}.$$

Fix $h \in \mathrm{H}$ arbitrarily and let

$$g = P_{\mathrm{G}_1}P_{\mathrm{G}_2}h = P_{\mathrm{G}_2}P_{\mathrm{G}_1}h.$$

By the first representation $g \in \mathrm{G}_1$ and, by the second, $g \in \mathrm{G}_2$. Hence $g \in \mathrm{G}_1 \cap \mathrm{G}_2$. If $h \in \mathrm{G}_1 \cap \mathrm{G}_2$, then $P_{\mathrm{G}_1}P_{\mathrm{G}_2}h = h$. Thus, one half of the theorem is proved. Now assume that P_{G_1} and P_{G_2} commute. Let

$$P_{\mathrm{G}_1}P_{\mathrm{G}_2} = P_{\mathrm{G}_2}P_{\mathrm{G}_1} = P.$$

[1] A geometrical implication of the commutativity of the operators P_{G_1} and P_{G_2} is that the subspaces $\mathrm{G}_1 \ominus (\mathrm{G}_1 \cap \mathrm{G}_2)$ and $\mathrm{G}_2 \ominus (\mathrm{G}_1 \cap \mathrm{G}_2)$ are orthogonal.

It follows that

$$P^2=(P_{G_1}P_{G_2})^2=P_{G_1}P_{G_2}P_{G_1}P_{G_2}=P_{G_1}P_{G_1}P_{G_2}P_{G_2}=P_{G_1}P_{G_2}=P$$

and

$$\begin{aligned}(Ph_1,h_2)&=(P_{G_1}P_{G_2}h_1,h_2)=(P_{G_2}h_1,P_{G_1}h_2)\\&=(h_1,P_{G_2}P_{G_1}h_2)=(h_1,P_{G_1}P_{G_2}h_2)=(h_1,Ph_2).\end{aligned}$$

These equations show that the operator $P=P_{G_1}P_{G_2}$ satisfies the conditions of the theorem of the preceding section. Therefore, it is a projection operator.

COROLLARY: *Two subspaces* G_1 *and* G_2 *are orthogonal if and only if*

$$P_{G_1}P_{G_2}=0.$$

THEOREM 2: *A finite sum of projection operators*

$$P_{G_1}+P_{G_2}+\ldots+P_{G_n}=Q \qquad (n<\infty)$$

is a projection operator if and only if

$$P_{G_i}P_{G_k}=0 \qquad (i\neq k)$$

i.e., if and only if the spaces G_j ($j=1,2,3,\ldots,n$) *are pairwise orthogonal. In this case*

$$Q=P_G,$$

where

$$G=G_1\oplus G_2\oplus\ldots\oplus G_n.$$

Proof: If the spaces G_j are pairwise orthogonal, then $Q^2=Q$, and, therefore, the sufficiency of the condition is evident. The last part of the assertion of the theorem is also evident. It remains only to prove the necessity of the condition. Let Q be a projection operator. Then

$$\|f\|^2\geqq(Qf,f)=\sum_{j=1}^{n}(P_{G_j}f,f)\geqq(P_{G_i}f,f)+(P_{G_k}f,f)$$

for any pair of distinct indices i and k. From this relation it follows that

$$\|P_{G_i}f\|^2+\|P_{G_k}f\|^2\leqq\|f\|^2.$$

In this inequality let

$$f=P_{G_k}h.$$

Then

$$\|P_{G_i}P_{G_k}h\|^2+\|P_{G_k}h\|^2\leqq\|P_{G_k}h\|^2$$

which yields

$$\|P_{G_i}P_{G_k}h\|=0$$

for $h \in \mathrm{H}$. Thus,

$$P_{\mathrm{G}_i} P_{\mathrm{G}_k} = 0$$

so that the spaces G_i and G_k are orthogonal.

THEOREM 3: *The difference of two projection operators,*

$$P_{\mathrm{G}_1} - P_{\mathrm{G}_2}, \tag{1}$$

is a projection operator if and only if $\mathrm{G}_2 \subset \mathrm{G}_1$. *In this case* $P_{\mathrm{G}_1} - P_{\mathrm{G}_2}$ *is the operator of projection on* $\mathrm{G}_1 \ominus \mathrm{G}_2$.

Proof: In view of the remark at the end of the preceding section, we attempt to find conditions for which the difference

$$Q = E - (P_{\mathrm{G}_1} - P_{\mathrm{G}_2})$$

is a projection operator. Since the equation

$$Q = (E - P_{\mathrm{G}_1}) + P_{\mathrm{G}_2}$$

represents Q as the sum of two projection operators, it follows from Theorem 2 that

$$(E - P_{\mathrm{G}_1}) P_{\mathrm{G}_2} = 0$$

or, equivalently,

$$P_{\mathrm{G}_2} = P_{\mathrm{G}_1} P_{\mathrm{G}_2}. \tag{2}$$

If $g \in \mathrm{G}_2$ then

$$g = P_{\mathrm{G}_2} g = P_{\mathrm{G}_1} P_{\mathrm{G}_2} g = P_{\mathrm{G}_1} g,$$

so that $g \in \mathrm{G}_1$. Since every element $g \in \mathrm{G}_2$ belongs to G_1, we have $\mathrm{G}_2 \subset \mathrm{G}_1$. This condition, which can be expressed in the form of (2), is necessary and sufficient in order that the difference (1) be a projection operator. It remains only to characterize the space G on which the operator (1) projects. The operator Q projects on

$$[\mathrm{H} \ominus \mathrm{G}_1] \oplus \mathrm{G}_2.$$

Hence, the operator (1) projects on

$$\mathrm{H} \ominus \{[\mathrm{H} \ominus \mathrm{G}_1] \oplus \mathrm{G}_2\}, \tag{3}$$

i.e., on the subspace of vectors orthogonal both to G_2 and to $\mathrm{H} \ominus \mathrm{G}_1$. Since this subspace consists of all the vectors of G_1 which are orthogonal to G_2, it is the subspace

$$\mathrm{G}_1 \ominus \mathrm{G}_2. \tag{4}$$

We notice that the difference (4) can be obtained directly from (3) by formally removing the brackets.

33. Monotone Sequences of Projection Operators

We shall prove that the relation $G_2 \subset G_1$ is equivalent to the inequality

$$(1) \qquad \| P_{G_2} f \| \leqq \| P_{G_1} f \|$$

for all $f \in H$. The inequality (1) is evidently equivalent to

$$(P_{G_2} f, f) \leqq (P_{G_1} f, f)$$

or

$$(\{P_{G_2} - P_{G_1}\} f, f) \leqq 0$$

for $f \in H$. The last two inequalities are generally expressed by

$$P_{G_2} \leqq P_{G_1}.$$

Thus, we wish to prove that the relation $G_2 \subset G_1$ is equivalent to the relation $P_{G_2} \leqq P_{G_1}$. This will permit us to introduce for consideration monotone sequences of projection operators.

First, let $G_2 \subset G_1$. Then it follows that

$$P_{G_2} = P_{G_2} P_{G_1}.$$

Therefore, for each $f \in H$,

$$P_{G_2} f = P_{G_2} P_{G_1} f$$

and

$$(2) \qquad \| P_{G_2} f \| \leqq \| P_{G_1} f \|.$$

Conversely, assume (2) for each $f \in H$. Consider

$$f = (E - P_{G_1}) h,$$

where h is an arbitrary element of H. From (2) and

$$P_{G_1}(E - P_{G_1}) h = 0,$$

we obtain

$$P_{G_2}(E - P_{G_1}) h = 0$$

or

$$P_{G_2} h = P_{G_2} P_{G_1} h.$$

Since this equality holds for each $h \in H$,

$$P_{G_2} = P_{G_2} P_{G_1},$$

so that $G_2 \subset G_1$. This completes the proof.

THEOREM: *If P_{G_k} $(k = 1, 2, 3, \ldots)$ is an infinite sequence of projection operators and if $P_{G_k} \leqq P_{G_{k+1}}$ $(k = 1, 2, 3, \ldots)$, then, as $k \to \infty$, P_{G_k} converges strongly to some projection operator P.*

Proof: For $m < n$ the difference $P_{G_n} - P_{G_m}$ is a projection operator. Therefore, for each $f \in H$,

$$(3) \quad \| P_{G_n} f - P_{G_m} f \|^2 = \| (P_{G_n} - P_{G_m}) f \|^2 = (\{P_{G_n} - P_{G_m}\} f, f) = \| P_{G_n} f \|^2 - \| P_{G_m} f \|^2.$$

Since, for fixed f, $\| P_{G_k} f \|^2$ increases with k but is bounded above by $\| f \|^2$, it has a finite limit. Hence, the right member of (3) tends to zero and the sequence $\{P_{G_n} f\}_{n=1}^{\infty}$ is fundamental in the sense of strong convergence. By the completeness of the space there exists the strong limit

$$f^* = \lim_{n \to \infty} P_{G_n} f.$$

We define the operator P by

$$f^* = Pf,$$

$f \in \mathrm{H}$. The operator P is obviously linear. Since

$$(P_{G_k} f, P_{G_k} g) = (P_{G_k} f, g) = (f, P_{G_k} g)$$

a passage to the limit yields

$$(Pf, Pg) = (Pf, g) = (f, Pg).$$

Therefore,

$$P = P^* = P^2,$$

so that P is a projection operator.

34. The Aperture of Two Linear Manifolds[2]

The present section is devoted to a concept which was introduced by B. Nagy and, independently of him, by M. G. Krein and M. A. Krasnoselski.[3]

DEFINITION: *The aperture of two linear manifolds in* H *is defined as the norm of the difference of the operators which project* H *on the closures of these two linear manifolds.*

The aperture of the linear manifolds M_1 and M_2 is denoted by the symbol $\Theta(\mathrm{M}_1, \mathrm{M}_2)$. Thus,

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) = \| P_1 - P_2 \| = \| P_2 - P_1 \|,$$

where P_1, P_2 are the operators of projection on the closed linear manifolds (subspaces) $\overline{\mathrm{M}}_1$, $\overline{\mathrm{M}}_2$, respectively. From the definition of aperture it follows that

$$\Theta(\mathrm{M}_1, \mathrm{M}_2) = \Theta(\overline{\mathrm{M}}_1, \overline{\mathrm{M}}_2) = \Theta(\mathrm{H} \ominus \mathrm{M}_1, \mathrm{H} \ominus \mathrm{M}_2).$$

Consider the identity

$$P_2 - P_1 = P_2(E - P_1) - (E - P_2) P_1.$$

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[2] The results of this paragraph are necessary only for the construction of the theory of symmetric extensions (Chapter 7).

[3] M. G. Krein and M. A. Krasnoselski [1], B. Sz. Nagy [2].