1 Let’s Make a Deal.

Years ago in America, there was a wildly popular game show on TV called *Let’s Make a Deal* with host Monty Hall. If you have seen this show, no description is needed; if you have never seen it, no accurate description is believable.

One of the games went as follows. Monty would be among the audience, questioning his current contestant. On the stage were three ‘doors’ or ‘curtains,’ labeled 1, 2 and 3. Monty would state, “Behind one door is a brand new car, but behind each of the other two doors is a goat.” (Note: For the sake of this discussion, a goat is a gag gift of no value while a car has immense value to the contestant.) “Select a door,” Monty continued, “and win the prize behind it.”

At this point, the contestant would select a door. For sake of this discussion, suppose that the contestant selects door 1. Now you might guess that door 1 would be opened and the contestant would leave with the gift revealed. But that would be wrong. Instead, Monty would have one of the remaining doors opened, let’s say door 3, and behind the door is revealed a goat.

Monty would then say to the contestant, “You may have the prize behind door 1 or you may switch and take the prize behind door 2. What will you do? Stay or switch?”

Since 1986 (this and the following biographical info are taken from Wikipedia), Marilyn vos Savant has written a column, called *Ask Marilyn* that appears in *Parade* magazine. Her credentials for having a column includes that she was listed in the *Guinness Book of World Records* under highest IQ. In addition, I have read her column more than a few times and have found her to be clever and entertaining.

In September, 1990, Marilyn analyzed the question I posed above. She stated that the better strategy is to switch. Many persons, including a number of math professors, wrote that she was wrong. In particular, the math professors stated that there was no advantage to either switching or staying. In this presentation, I will take the math professors’ advice to be to stay. They can’t possibly object to my doing this b/c their position is that it does not matter what the contestant does.

I will analyze this question b/c its solution reveals several important ideas that will recur in this course.

First of all, upon reflection you can see that there is no really good answer to this question. This is not an arithmetic question like, “What is the square of 5?” which has a unique exact correct answer, 25.

By contrast, both Marilyn’s and the math professors’ answers are flawed. Marilyn tells you to switch, but the car might be behind door 1. The math professors say to stay, but the car might be behind door 2.

It reminds me of a friend of mine who had the following message taped to his office door:

Advice for playing the stock market: Buy low and sell high. If the stock does not increase in value, then don’t buy it.

Thus, each strategy, stay or switch, might win or it might lose. As a result, we adopt the following approach, which is one of the big ideas of probability and statistics:

We evaluate a strategy by seeing how it performs in the long-run.
I will begin by analyzing the math professors’ reasoning. There are three possible states of nature, and they are presented in the display below.

<table>
<thead>
<tr>
<th>State</th>
<th>Door 1</th>
<th>Door 2</th>
<th>Door 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Car</td>
<td>Goat</td>
<td>Goat</td>
</tr>
<tr>
<td>B</td>
<td>Goat</td>
<td>Car</td>
<td>Goat</td>
</tr>
<tr>
<td>C</td>
<td>Goat</td>
<td>Goat</td>
<td>Car</td>
</tr>
</tbody>
</table>

Their reasoning is outlined in the following steps.

1. From the perspective of the contestant, States A, B and C are equally likely to be true.
2. Monty selects door 3 at random, opens it, and reveals a goat.
3. Monty’s data proves that State C is not true. The remaining States, A and B are still equally likely to be true.
4. Thus, there is no advantage to either staying or switching.

The mathematicians made two incredibly serious errors. And the amazing fact is that so many of them made these two errors without seemingly being aware of the fact!

Their first error is that they seemed to automatically assume that Monty selected a door to open at random. Now, I am not surprised that they all made this assumption. Go to any probability textbook and read the exercises. I conjecture that you will find that nearly all problems involve wording of the type, “Sally selects ... at random.” This is what mathematicians are good at. You give them a problem where Sally selects at random and there is a strong possibility that the mathematicians will solve it. If you give them a problem in which something was selected with any method other than at random, and there is a strong possibility they won’t make any progress towards a solution.

This reminds me of two of my favorite sayings, although I am not really sure how well they apply:

If the only tool you have is a hammer, the whole world looks like a nail.

The world’s greatest swordsman will not be beaten by the world’s second greatest swordsman; he will be beaten by someone who picks up a sword for the first time.

The mathematicians second error was that they never investigated whether their assumption makes sense in the real world. If, indeed, Monty selected a door at random, it stands to reason (and probability theory supports this) that occasionally he would have opened a door and revealed the car. I admit I watched the show a few times in my life; I know people who watched it many times; and I have read many comments on the show. To my knowledge, it has never been suggested that Monty ever opened a door to reveal the car. Every time Monty opened a door he revealed a goat.

The inescapable conclusion is that Monty always knew where the car was and he deliberately avoided revealing the car. Also, it appears that Monty always opened a door when a contestant was playing this game.
Marilyn’s answer reflected her knowledge of what was the correct assumption to make for this problem. There are several ways to see why switching is better than staying, but here is the slickest way; well, at least in my opinion.

Here is a better way to think about what Monty was doing: whatever door the contestant selects, Monty offers a trade of the selected door for the remaining two doors, with the proviso that if the other two doors both have goats, the contestant gets only one goat and if the other two doors have one goat and the car, the contestant gets only the car. With this perspective, Marilyn argued that by switching the chance of winning the car increases from one-in-three to two-in-three.

But why should the contestant prefer two-in-three over one-in-three? Watch poker on TV; you will see that quite often one-in-three defeats two-in-three. Well, Marilyn and the probabilists agree on the answer to this question: B/c in the long-run two-in-three wins more times than one-in-three.

Now, before you get too smug in your belief that Marilyn was correct, remember: the contestant only played the game once. Yes, if the contestant was allowed to play 1000 times, and must select between the strategies: always stay and always switch, then he/she should definitely choose the strategy always switch. But there is no way to **prove** that it is better to switch with only one play.

And, finally, this brings us to the one big assumption that both Marilyn and the mathematicians make: They all assume that the contestant was guessing when he/she selected door 1. There is no way of knowing that the contestant was guessing. As a statistician, I know that it is impossible to distinguish between guessing and not guessing based on data from one trial. Yes, if we had 1,000 trials I can, in a very specific way that I cannot detail now, determine statistically whether the contestant is better than guessing. But even this study would be flawed; unless we are willing to give away hundreds of new cars, we can’t really test the contestant. Perhaps his/her ESP surfaces only in the presence of a large reward.

Now, I admit that I personally believe that for any particular contestant, switching is better than staying. Either he/she is guessing or the ‘amount of ESP’ is not enough to the two-doors-to-one-door advantage inherent in switching.

I doubt that the contestant has ESP b/c I can’t imagine a mechanism to explain it. But how compelling is this argument? Well, perhaps not so much, b/c I note two things:

- Throughout the human experience, people have been killed by things—bacteria, radiation, viruses—which they did not even suspect existed.

- Billions of people have belief structures that others find puzzling and without merit or explanation.

In short, here is my message. All analyses involve the making of assumptions. The worst error is to be unaware of the assumptions you are making. Once you identify your assumptions, try to determine whether they are reasonable.