

BOOTSTRAPPED CONFIDENCE BANDS FOR PERCENTILE LIFETIME

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Summary

We propose bootstrapped confidence bands for the percentile lifetime function. Our method is based on a joint approximation of the empirical and quantile processes and their bootstrapped counterparts. Modest simulations support the results. Confidence bands are applied to quantile and median residual lifetimes of tractor rear brakes.

1. Introduction

Let F be a life distribution function with support $(0, T_F)$, where $T_F = \sup\{t > 0: F(t) < 1\} \leq \infty$ and corresponding quantile function $Q(x) = \inf\{t > 0: F(t) \geq x\}$, $0 < x < 1$, $Q(0) = 0$, $Q(1) = T_F$. Let $0 < p < 1$ be any fixed number, and consider the $1-p$ percentile residual lifetime

$$R(t, p) = Q(1 - p(1 - F(t))) - t, \quad 0 \leq t < \infty,$$

originally introduced by Haines and Singpurwalla [9], and interpreted as the $1-p$ percentile additional time to failure given no failure by time t . Schmittlein and Morrison [12] explain in detail the potential advantages of the median residual lifetime $R(t, 1/2)$ over the more frequently used mean residual lifetime. These two papers deal with characterization theorems, while Gupta and Langford [8] provides results for the associated inversion problem. Joe and Proschan [10], [11] use $R(t, p)$ to compare two classes of life distributions.

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Let X_1, \dots, X_n be independent, identically distributed random variables with common distribution function F . The order statistics of X_1, \dots, X_n will be denoted by $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$. Besides the empirical distribution function

$$F_n(t) = (1/n) \# \{1 \leq i \leq n : X_i \leq t\},$$

we consider also the empirical quantile function

$$Q_n(x) = \begin{cases} 0, & x = 0, \\ X_{k,n}, & (k-1)/n < x \leq k/n, 1 \leq k \leq n. \end{cases}$$

Csörgő and Csörgő [4] introduced the following estimate of $R(t, p)$:

$$R_n(t, p) = Q_n(1 - p(1 - F_n(t))) - t, \quad 0 < p < 1, 0 \leq t < \infty.$$

Let

$$r_n(t, p) = n^{1/2}(R_n(t, p) - R(t, p)).$$

Csörgő and Csörgő [4] studied the asymptotic properties of the process $f(R(t, p) + t)r_n(t, p)$, where $f = F'$ is the density function of F . Assuming more regularity conditions on F than we are going to here, they proved strong approximations of $f(R(t, p) + t)r_n(t, p)$ by Gaussian processes, and proposed asymptotic confidence bands for $R(t, p)$. While the limit process of $f(R(Q(x), p) + Q(x))r_n(Q(x), p)$ does not depend on F (distribution free), they did not attempt to compute the distributions of the supremums of the limit process over appropriate intervals such as $t \in [0, T]$, p fixed, or $(t, p) \in [0, T] \times [a, b]$, with $F(T) < 1$ and $0 < a \leq b < 1$. Thus their method gives asymptotic confidence bands for $R(t, p)$ only in principle.

The main aim of this note is to show that the bootstrap method can be used to construct confidence bands for $R(t, p)$. The confidence bands are applied to lifetimes of tractor rear brakes following a modest simulation.

2. Bootstrapped empirical and quantile functions

Given X_1, \dots, X_n , let Y_1, \dots, Y_m be conditionally independent random variables with common distribution function F_n . The empirical distribution function of Y_1, \dots, Y_m ,

$$F_{m,n}(t) = (1/m) \# \{1 \leq i \leq m : Y_i \leq t\},$$

is called the bootstrapped empirical distribution function of F_n . Let $Y_{1,m} \leq \dots \leq Y_{m,m}$ be the ordered sample of Y 's and consider the bootstrapped empirical quantile function

$$Q_{m,n}(x) = \begin{cases} 0, & x=0, \\ Y_{k,m}, & (k-1)/m < x \leq k/m, 1 \leq k \leq m. \end{cases}$$

Using these bootstrapped functions we can define the bootstrapped empirical percentile residual lifetime as

$$(2.1) \quad R_{m,n}(t, p) = Q_{m,n}(1 - p(1 - F_{m,n}(t))) - t, \quad 0 < p < 1, 0 \leq t < \infty$$

and the bootstrapped version of r_n defined as

$$(2.2) \quad r_{m,n}(t, p) = m^{1/2}(R_{m,n}(t, p) - R_n(t, p)).$$

The following theorem provides strong approximations for r_n and $r_{m,n}$ which will be used for construction of confidence bands for $R(t, p)$.

THEOREM 2.1 *We assume the conditions*

- (i) *F has a density function $f = F'$ which is positive and continuous over the support $(0, T_F)$ of F,*
- (ii) *there are two positive constants c and C such that $c \leq m/n \leq C$ for each m and n .*

If our probability space is rich enough, we can define two independent sequences of two-parameter Gaussian processes $\{\Gamma_n(t, p), t \geq 0, 0 < p < 1\}_{n=1}^\infty$ and $\{\Delta_m(t, p), t \geq 0, 0 < p < 1\}_{m=1}^\infty$ such that

$$(2.3) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |r_n(t, p) - \Gamma_n(t, p)| = o_P(1), \quad n \rightarrow \infty,$$

and

$$(2.4) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |r_{m,n}(t, p) - \Delta_m(t, p)| = o_P(1), \quad m, n \rightarrow \infty,$$

where $0 < a \leq b < 1$ and $T < T_F$. We have in addition that

$$\{\Gamma_n(t, p), t \geq 0, 0 < p < 1\} \stackrel{D}{=} \{\Delta_m(t, p), t \geq 0, 0 < p < 1\},$$

for each m, n , and $E \Gamma_n(t, p) = E \Delta_m(t, p) = 0$,

$$\begin{aligned} E \Gamma_n(t, p) \Gamma_n(s, q) &= E \Delta_m(t, p) \Delta_m(s, q) \\ &= \frac{p+q - pq(1 - F(t \wedge s)) - (p(1 - F(t)) \vee q(1 - F(s)))}{f(Q(1 - p(1 - F(t))))f(Q(1 - q(1 - F(s))))} \\ &\quad - \frac{p(F(t) \wedge (1 - q(1 - F(s)))) + q(F(s) \wedge (1 - p(1 - F(t))))}{f(Q(1 - p(1 - F(t))))f(Q(1 - q(1 - F(s))))} \end{aligned}$$

where $t \wedge s = \min(t, s)$ and $t \vee s = \max(t, s)$.

The next theorem will allow us to use the bootstrapped maximal deviation statistic to derive confidence bands.

THEOREM 2.2. *Assuming (i) of Theorem 2.1 we have that*

$$G(y) = G(y, a, b, T) = P \left\{ \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |\Gamma_n(t, p)| < y \right\} .$$

is continuous in $y \in (0, \infty)$ for each $0 < a \leq b < 1$ and $T < T_F$.

Now we are in the position to construct our promised bootstrapped confidence bands. Using our bootstrap generation of processes $r_{m,n}^{(i)}$, $1 \leq i \leq N$, N times, by the Glivenko-Cantelli theorem we get

$$(2.5) \quad G_{N,m,n}(x) = (1/N) \# \{ 1 \leq i \leq N : \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |r_{m,n}^{(i)}(t, p)| \leq x \} \\ \rightarrow P \left\{ \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |r_{m,n}(t, p)| \leq x \right\} \text{ a.s.}$$

and uniformly in x for n and m fixed. By Theorem 2.1 and (2.5) we get

$$(2.6) \quad G_{N,m,n}(x) \rightarrow G(x) \text{ a.s.}$$

and uniformly in x as $N, m, n \rightarrow \infty$. Let $\alpha \in (0, 1)$ be fixed, and define

$$c_{N,m,n}(\alpha) = \inf \{ x > 0 : G_{N,m,n}(x) \geq 1 - \alpha \}$$

and

$$c(\alpha) = \inf \{ x > 0 : G(x) \geq 1 - \alpha \} .$$

By Theorem 2.2 and an argument like that of Corollary 17.3 in Csörgő et al. [5] we get that

$$(2.7) \quad c_{N,m,n}(\alpha) \rightarrow c(\alpha) \text{ a.s.}$$

as $N, m, n \rightarrow \infty$. Hence from (2.7) we see that an asymptotically $(1 - \alpha)100\%$ confidence band for $R(t, p)$ is given by

$$(2.8) \quad R_n(t, p) - c_{N,m,n}(\alpha)n^{-1/2} \leq R(t, p) \leq R_n(t, p) + c_{N,m,n}(\alpha)n^{-1/2} , \\ 0 \leq t \leq T, a \leq p \leq b .$$

If we fix a $p_0 = a = b$, $0 < p_0 < 1$, then (2.8) reduces to a confidence band for $R(t, p_0)$.

3. Applications

Simulations were carried out for exponential data with sample sizes of $n = 10, 20, 50, 100$, and 500 . Bootstrap samples of size $m = n$ were drawn $N = 1000$ times, and the maximal deviations of the bootstrapped percentile lifetime process (2.2) were recorded. We computed the maximal deviations over a finite interval $[0, T_{0.9}]$, with $T_{0.9} = \sup \{ t > 0 : F(t) < 0.9 \}$. The bootstrapped empirical distribution, $G_{N,m,n}(y, 0.5)$ defined in (2.5), appeared stable for $n = m \geq 50$ (see Fig. 1). Of course, a heavier tailed distribution might require further truncation for stable behavior.

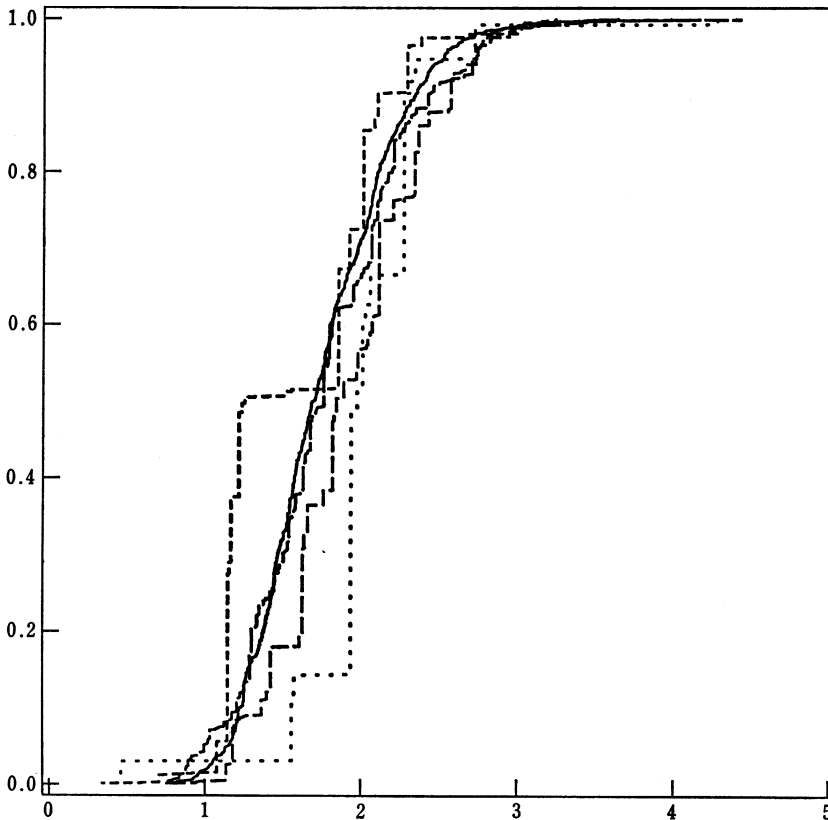


Fig. 1. Simulations of $G_{N,m,n}(y, 0.5)$ with $N=1000$ and $m=n=10$ (dot), 20 (short dash), 50 (medium dash), 100 (long dash) and 500 (solid).

We examined 107 failure times for right rear breaks on D9G-66A Caterpillar tractors. These data are available in Barlow and Campo [1] and in Doksum and Yandell [7], where total time on test and other tests of exponentiality were investigated. Here we present (asymptotically) 50% and 90% confidence bands based on (2.8) for the quartiles (25th and 75th percentile) and the median. The critical values were taken from the bootstrapped empirical distribution $G_{1000,107,107}(y, p)$, $p=0.25, 0.5, 0.75$, with the supremum taken over $[0, 2900]$, which corresponds roughly to the interval $\{t>0: F_n(t)<0.8\}$. The confidence bands are somewhat wide, reflecting the heavy tail of the failure distribution as noted by earlier authors. Moving the truncation point lower tightens the bands slightly, but one can still have long percentile lifetimes.

Some simulations were done varying the bootstrap sample size m separately from n . The empirical distributions were largely unaffected when $n \geq 100$. This suggests that one could use $m=100$ for large samples. The computing was very fast, taking less than 5 minutes real

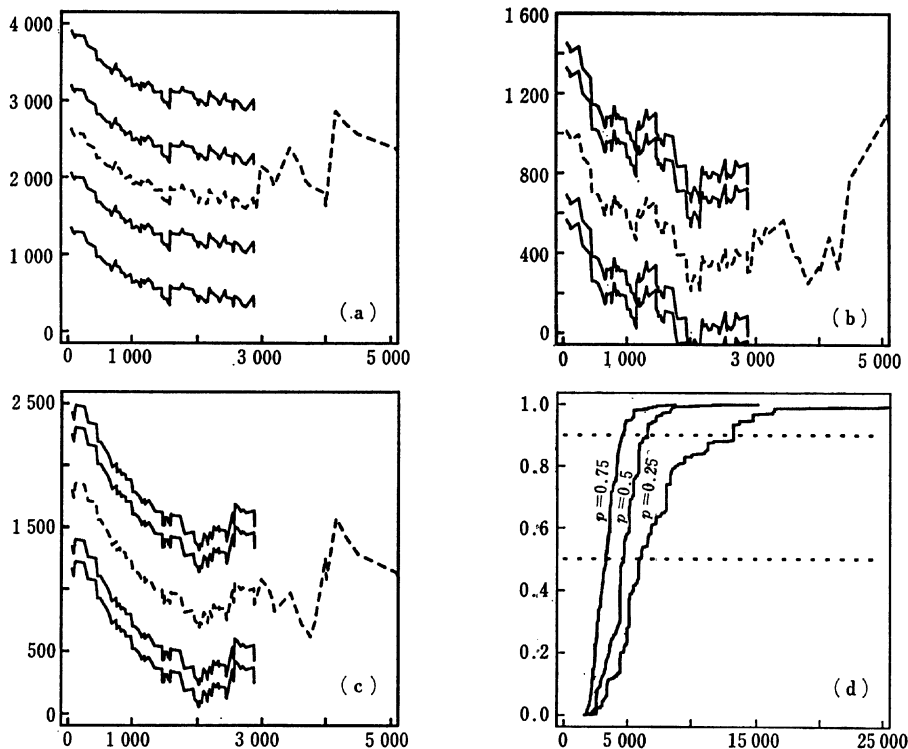


Fig. 2. Percentile lifetime of tractor rear brakes: (a) confidence bands for $R(t, 0.75)$ on $[0, 2900]$; (b) confidence bands for $R(t, 0.5)$ on $[0, 2900]$; (c) confidence bands for $R(t, 0.25)$ on $[0, 2900]$; (d) bootstrapped empirical distribution $G_{1000, 107, 107}(y, p)$ for $p = 0.25, 0.5, 0.75$.

time on the U. Wisconsin Statistics Research Computer, a VAX 11/750 with floating point, for $n = m = 500$ with $N = 1000$ bootstrap trials.

4. Proofs

Introduce the processes

$$\alpha_n(t) = n^{1/2}(F'_n(t) - F(t)), \quad 0 \leq t < \infty,$$

$$u_n(x) = n^{1/2}(x - F(Q_n(x))), \quad 0 \leq x \leq 1,$$

and their bootstrapped versions

$$\alpha_{m,n}(t) = m^{1/2}(F'_{m,n}(t) - F'_n(t)), \quad 0 \leq t < \infty,$$

$$u_{m,n}(x) = m^{1/2}(F(Q_n(x)) - F(Q_{m,n}(x))), \quad 0 \leq x \leq 1,$$

Bickel and Freedman [2] and Csörgő et al. [5] proved the following theorem.

THEOREM 4.1. *We assume that condition (ii) of Theorem 2.1 holds. We can define two independent sequences of Brownian bridges $\{B_n(x), 0 \leq x \leq 1\}_{n=1}^\infty$ and $\{D_m(x), 0 \leq x \leq 1\}_{m=1}^\infty$ such that*

$$(4.1) \quad \sup_{0 \leq t < \infty} |\alpha_n(t) - B_n(F(t))| = o_P(1), \quad n \rightarrow \infty,$$

$$(4.2) \quad \sup_{0 \leq x \leq 1} |u_n(x) - B_n(x)| = o_P(1), \quad n \rightarrow \infty,$$

and

$$(4.3) \quad \sup_{0 \leq t < \infty} |\alpha_{m,n}(t) - D_m(F(t))| = o_P(1), \quad m, n \rightarrow \infty,$$

$$(4.4) \quad \sup_{0 \leq x \leq 1} |u_{m,n}(x) - D_m(x)| = o_P(1), \quad m, n \rightarrow \infty.$$

Now we are ready to give the

PROOF OF THEOREM 2.1. Let

$$F_n(t, p) = \frac{pB_n(F(t)) - B_n(1 - p(1 - F_n(t)))}{f(Q(1 - p(1 - F(t))))}.$$

We write

$$(4.5) \quad r_n(t, p) = n^{1/2} \{Q_n(1 - p(1 - F_n(t))) - Q(1 - p(1 - F_n(t)))\} \\ + n^{1/2} \{Q(1 - p(1 - F_n(t))) - Q(1 - p(1 - F(t)))\}.$$

Using a one term Taylor expansion we get

$$n^{1/2} \{Q(F(Q_n(x))) - Q(x)\} = \frac{-1}{f(Q(x))} u_n(x) + \left\{ \frac{1}{f(Q(x))} - \frac{1}{f(Q(\tau_n(x)))} \right\} u_n(x),$$

where $F(Q_n(x)) \wedge x \leq \tau_n(x) \leq F(Q_n(x)) \vee x$. Using condition (i), Theorem 4.1, and the fact that $\sup_{0 \leq x \leq 1} |\tau_n(x) - x| = o_P(1)$, we obtain that

$$(4.6) \quad \sup_{\varepsilon \leq x \leq 1-\varepsilon} \left| n^{1/2} \{Q_n(x) - Q(x)\} + \frac{B_n(x)}{f(Q(x))} \right| = o_P(1),$$

for each $0 < \varepsilon < 1/2$. The Glivenko-Cantelli theorem gives

$$(4.7) \quad \sup_{0 \leq t < \infty} |F_n(t) - F(t)| = o_P(1),$$

and therefore

$$(4.8) \quad \sup_{\alpha \leq p \leq \beta} \sup_{0 \leq t \leq T} \left| n^{1/2} \{Q_n(1 - p(1 - F_n(t))) - Q(1 - p(1 - F_n(t)))\} \right. \\ \left. + \frac{B_n(1 - p(1 - F(t)))}{f(Q(1 - p(1 - F(t))))} \right| = o_P(1)$$

by the almost sure continuity of the Brownian bridge process. By the mean value theorem we have

$$n^{1/2} \{Q(1-p(1-F_n(t))) - Q(1-p(1-F(t)))\} \\ = \frac{p\alpha_n(t)}{f(Q(1-p(1-F(t))))} + \left\{ \frac{1}{f(Q(\eta_n(t)))} - \frac{1}{f(Q(1-p(1-F(t))))} \right\} p\alpha_n(t),$$

where $1-p(1-(F_n(t) \vee F(t))) \leq \eta_n(t) \leq 1-p(1-(F_n(t) \wedge F(t)))$. Using again (4.7), condition (i) and Theorem 4.1, we get

$$(4.9) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} \left| n^{1/2} \{Q(1-p(1-F_n(t))) - Q(1-p(1-F(t)))\} - \frac{pB_n(F(t))}{f(Q(1-p(1-F(t))))} \right| = o_P(1).$$

Collecting now (4.5), (4.8), and (4.9) we obtain (2.3).

The proof of (2.4) follows similar lines. Toward this end let

$$A_m(t, p) = \frac{pD_m(F(t)) - D_m(1-p(1-F(t)))}{f(Q(1-p(1-F(t))))}.$$

We write again

$$(4.10) \quad r_{m,n}(t, p) = m^{1/2} \{Q_{m,n}(1-p(1-F_{m,n}(t))) - Q_n(1-p(1-F_{m,n}(t)))\} \\ + m^{1/2} \{Q_n(1-p(1-F_{m,n}(t))) - Q_n(1-p(1-F_n(t)))\}.$$

Just like before, we arrive at

$$m^{1/2} \{Q_{m,n}(x) - Q_n(x)\} = \frac{-1}{f(Q(x))} u_{m,n}(x) + \left\{ \frac{1}{f(Q(x))} - \frac{1}{f(Q(\tau_{m,n}(x)))} \right\} u_{m,n}(x),$$

where $F(Q_{m,n}(x) \wedge Q_n(x)) \leq \tau_{m,n}(x) \leq F(Q_{m,n}(x) \vee Q_n(x))$. Using again Theorem 4.1 and its consequence that

$$\sup_{0 \leq x \leq 1} |F(Q_{m,n}(x)) - x| = o_P(1), \quad \sup_{0 \leq x \leq 1} |F(Q_n(x)) - x| = o_P(1),$$

and

$$(4.11) \quad \sup_{0 \leq t < \infty} |F_{m,n}(t) - F(t)| = o_P(1),$$

we obtain

$$(4.12) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} \left| m^{1/2} \{Q_{m,n}(1-p(1-F_{m,n}(t))) - Q_n(1-p(1-F_{m,n}(t)))\} + \frac{D_m(1-p(1-F(t)))}{f(Q(1-p(1-F(t))))} \right| = o_P(1).$$

The estimation of the second term on the right side of (4.10) requires somewhat more calculations. First we observe that

$$(4.13) \quad \left| m^{1/2} \{Q_n(1-p(1-F_{m,n}(t))) - Q_n(1-p(1-F_n(t)))\} \right|$$

$$\begin{aligned}
 & \left| -\frac{pD_m(F(t))}{f(Q(1-p(1-F(t))))} \right| \\
 \leq & \left| m^{1/2} \{Q(1-p(1-F_{m,n}(t))) - Q(1-p(1-F_n(t)))\} \right. \\
 & \left. -\frac{pD_m(F(t))}{f(Q(1-p(1-F(t))))} \right| \\
 & + \left(\frac{m}{n} \right)^{1/2} \left| n^{1/2} \{Q_n(1-p(1-F_n(t))) - Q(1-p(1-F_n(t)))\} \right. \\
 & \left. -\frac{B_n(t)}{f(Q(1-p(1-F(t))))} \right| \\
 & + \left(\frac{m}{n} \right)^{1/2} \left| n^{1/2} \{Q_n(1-p(1-F_{m,n}(t))) - Q(1-p(1-F_{m,n}(t)))\} \right. \\
 & \left. -\frac{B_n(t)}{f(Q(1-p(1-F(t))))} \right| \\
 = & a_n^{(1)}(t, p) + a_n^{(2)}(t, p) + a_n^{(3)}(t, p) .
 \end{aligned}$$

By the mean value theorem and Theorem 4.1 we get

$$(4.14) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |a_n^{(3)}(t, p)| = o_P(1) .$$

From (4.8) and condition (ii) we immediately obtain that

$$(4.15) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |a_n^{(2)}(t, p)| = o_P(1) .$$

The Glivenko-Cantelli theorem of (4.11) implies that we can write $F_{m,n}$ instead of F_n in (4.8), and therefore

$$(4.16) \quad \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |a_n^{(3)}(t, p)| = o_P(1) .$$

Thus (4.12)–(4.16) give (2.4). By the representation of Γ_n and Δ_m in terms of Brownian bridges the calculation of their covariances is only an exercise.

PROOF OF THEOREM 2.2. We prove that

$$G(y) = P \left\{ \sup_{a \leq p \leq b} \sup_{0 \leq t \leq T} |\Gamma(t, p)| \leq y \right\}$$

is continuous in $y \in (0, \infty)$, where

$$\Gamma(t, p) = \frac{pB(F(t)) - B(1-p(1-F(t)))}{f(Q(1-p(1-F(t))))} ,$$

and B is a Brownian bridge process. It is clear the Γ is a representation for both Γ_n and Δ_m . Let $\varepsilon, \delta > 0$ be arbitrary numbers. Using the uniform continuity of $1/f(Q(u))$ on every closed interval of $(0, 1)$ and Lemma 1.1.1 of Csörgő and Révész [6] (cf. Theorem 2.C in Burke

et al. [3]) we can define the finite sequences $a=p_1 < p_2 < \dots < p_K=b$ and $0=t_1 < \dots < t_M=T$ such that

$$P \left\{ \max_{1 \leq i \leq K-1} \max_{1 \leq j \leq M-1} \sup_{p_i \leq p \leq p_{i+1}} \sup_{t_j \leq t \leq t_{j+1}} |\Gamma(t, p_i) - \Gamma(t, p)| > \varepsilon \right\} \leq \delta.$$

The random variables $\xi_{i,j} = \Gamma(t_j, p_i)$ have Gaussian joint distributions, $E\xi_{i,j} = 0$ and $E\xi_{i,j}^2 > 0$, $1 < i < K$, $1 < j < M$. By Theorem 1 of Tsirel'son [13] we know that the distribution function of $\max_{1 \leq i \leq K-1} \max_{1 \leq j \leq M-1} |\xi_{i,j}|$ is continuous on $(0, \infty)$, and hence we have also proved Theorem 2.2.

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