ASYMPTOTIC THEORY OF SOME BOOTSTRAPPED EMPIRICAL PROCESSES

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We develop an asymptotic theory of bootstrapped weighted empirical and quantile processes. Utilizing this theory, we present a general body of techniques to establish the asymptotic validity of the bootstrap method of constructing confidence bands for some statistical functions. These techniques are demonstrated to be applicable to the construction of bootstrap confidence bands for such interesting statistical functions as the mean residual life function, the total time on test transform and the Lorenz curve.

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1. Introduction. Efron (1979) introduced the bootstrap method of constructing confidence intervals for a real valued population parameter $\theta(F)$. Given independent observations $X_1,\ldots,X_n$ from F, this method consists of approximating the sampling distribution of an appropriate parameter estimator $\hat{\theta}_n = \hat{\theta}_n(X_1,\ldots,X_n)$ of $\theta(F)$ by means of the sampling distribution of the quantities $\hat{\theta}_{m,n} = \hat{\theta}_m(\tilde{X}_1,\ldots,\tilde{X}_m)$, where the $m \geq 1$ observations $\tilde{X}_1,\ldots,\tilde{X}_m$ are sampled independently with distribution function $F_n(x) = n^{-1}\#\{k : 1 \leq k \leq n,\ X_k \leq x\}$, $-\infty < x < \infty$. For a clear account of the method and its relationship to other resampling methods we refer to Efron (1982). Bickel and Freedman (1981) showed that the bootstrap approximation is asymptotically valid for many statistics of interest such as the sample mean and variance, t-statistics, general U-statistics or even more general von Mises functionals of F_n, and for certain types of L-statistics. Singh (1981) studied the accuracy of the bootstrap approximation for the sample mean and sample quantiles.

Bickel and Freedman (1981) also established the weak convergence of the bootstrapped version of the empirical process and the same for the bootstrapped general quantile process on a restricted interval. From these results they were able to deduce the asymptotic validity of the bootstrap method of forming confidence *bands* for the true distribution function F, and also for its quantile function

(1.1) $$Q(s) = \inf\{x : F(x) \geq s\},\ 0 < s \leq 1,\ Q(0) = Q(0+),$$

on a proper subinterval of [0,1] not containing the endpoints. Shorack (1982) gave a simple proof of the weak convergence of the bootstrapped empirical process.

In the present paper we pursue further the second line of the Bickel and Freedman (1981) study and consider the validity of the bootstrap for general empirical functions containing as special cases the empirical distribution function and the empirical quantile function. We now introduce some notations.

Let $R_F(\cdot)$ be a statistical function of interest defined on an interval $I \subseteq R$, and let $R_n(\cdot) = R_n(\cdot;X_1,\ldots,X_n)$ be an appropriate estimator of $R_F(\cdot)$ on I (see the examples below). Typically, for the process

$$r_n(\cdot) = n^{\frac{1}{2}}(R_n(\cdot)-R_F(\cdot)) \tag{1.2}$$

one can find a sequence of copies $G_F^{(n)}(\cdot)$ of a Gaussian process $G_F(\cdot)$ on I, i.e., $\{G_F^{(n)}(t) : t \in I\} \overset{\mathcal{D}}{=} \{G_F(t) : t \in I\}$ for each $n \geq 1$, such that

$$\sup_{t\in I} |r_n(t) - G_F^{(n)}(t)| \to 0 \quad \text{in probability as} \quad n \to \infty. \tag{1.3}$$

(Here and in what follows $\overset{\mathcal{D}}{=}$ stands for the equality of all finite dimensional distributions of the stochastic processes on the two sides.) Consequently, given $0 < \alpha < 1$, we have

$$P\{R_n(t)-cn^{-\frac{1}{2}} \leq R_F(t) \leq R_n(t) + cn^{-\frac{1}{2}},\ t \in I\} \to 1-\alpha, \text{ as } n \to \infty,$$

provided that $G_F(c) = 1-\alpha$ and $c = c(\alpha,F)$ is a continuity point of the distribution function

$$G_F(x) = P\{\sup_{t\in I} |G_F(t)| \leq x\},\quad x \geq 0. \tag{1.4}$$

This means that $\{R_n(t) \pm cn^{-\frac{1}{2}},\ t \in I\}$ is an asymptotically correct $(1-\alpha)100\%$ confidence band for the statistical function R_F.

It is rare that this method of forming asymptotically correct confidence bands is feasible, since there are only a few cases when $c = c(\alpha,F)$ is independent of F and its analytical form is known. The most well-known case when this is true is the choice $R_F = F$, $R_n = F_n$ and F is continuous. In this case G_F is the distribution function of

$$(1.5) \qquad \sup_{-\infty<x<\infty} |B(F(x))| = \sup_{0\leq s\leq 1} |B(s)|,$$

where B is a Brownian bridge.

Consider the bootstrapped version of the empirical function $R_n(\cdot)$ given by

$$\tilde{R}_{m,n}(\cdot) = R_m(\cdot;\, \tilde{X}_1,\ldots,\tilde{X}_m).$$

Suppose we were able to show that there is a sequence of Gaussian processes such that $\{\tilde{G}_F^{(m)}(t) : t \in I\} = \{G_F(t) : t \in I\}$ for each m and

$$(1.6) \qquad \sup_{t\in I} |m^{\frac{1}{2}}(\tilde{R}_{m,n}(t)-R_n(t)) - \tilde{G}_F^{(m)}(t)| = o_P(1)$$

as $n \to \infty$, where $m = m(n) \to \infty$ at an appropriate rate. From (1.6) we can conclude that whenever x is a continuity point of G_F in (1.4) then

$$(1.7) \qquad G_{m,n}(x) = P\{\sup_{t\in I} m^{\frac{1}{2}}|\tilde{R}_{m,n}(t) - R_n(t)| \leq x\} \longrightarrow G_F(x)$$

for the same $m = m(n)$ sequence as $n \to \infty$. Now fix $0 < \alpha < 1$ and suppose we can show that

$$(1.8) \qquad c = c(\alpha,F) = \inf\{x : G_F(x) \geq 1-\alpha\} \text{ is a continuity point of } G_F.$$

Generating now the bootstrapped $\tilde{R}_{m,n}$ function N times independently: $\tilde{R}^{(i)}_{m,n}$, $1 \leq i \leq N$, on using the Glivenko-Cantelli theorem it can be easily verified that

$$(1.9) \qquad G_{N,m,n}(x) = \frac{1}{N}\#\{1 \leq i \leq N: m^{\frac{1}{2}} \sup_{t\in I} |\tilde{R}^{(i)}_{m,n}(t) - R_n(t)| \leq x\}$$

$$\longrightarrow G_{m,n}(x) \quad \text{a.s.}$$

uniformly in x as $N \to \infty$ and n,m are fixed. We define

$$c_{N,m,n} = c_{N,m,n}(\alpha) = \inf\{x: G_{N,m,n}(x) \geq 1-\alpha\}.$$

By (1.7) and (1.9) one easily obtains

$$(1.10) \quad P\{\sup_{t\in I} n^{\frac{1}{2}} |R_n(t) - R_F(t)| \leq c_{N,m,n}\} \longrightarrow 1-\alpha ,$$

as $N,m,n \to \infty$.

The purpose of this paper is to provide some techniques that should prove useful in establishing (1.6) for a variety of empirical functions of statistical interest. We will demonstrate how to apply our techniques by showing the validity of (1.6) for the empirical total time on test transform, the empirical mean residual life function, and the empirical Lorenz curve. The first two play important roles in reliability and survival analysis and the third is a basic empirical tool in economic concentration theory.

The philosophy of the bootstrap principle includes the appealing heuristic idea that bootstrapped versions $r_{m,n}$ of processes r_n behave asymptotically the same way as the original processes r_n. This is indeed the case for the three examples we consider. The validity of of the bootstrap will be proven under exactly the same optimal conditions under which the weak convergence result in (1.3) has been previously established by M. Csörgő, S. Csörgő and L. Horváth [Cs-Cs-H] (1986) for the three processes r_n corresponding to these empirical functions.

Our results for these three empirical functions are stated as theorems in the next section. In Section 3 we apply these results for constructing confidence bands for the total time on test transform of tractor rear brakes and present a simulation study of the accurary of the bootstrap. The necessary technical tools are detailed and proven in Section 4, and the proofs for the theorems in Section 2 are provided in Section 5.

2. Bootstrapped mean residual life, total time on test, and Lorenz processes. Throughout this section and Section 5 we assume that the random variable (rv) X, for which $F(x) = \Pr\{X \le x\}$, $-\infty < x < \infty$, is nonnegative, i.e., $F(0) = 0$.

2.1. Our first example of a statistical function R_F is the mean residual life function

$$M_F(t) = E(X-t \mid X>t)$$

$$= \int_t^\infty (1-F(x))dx/(1-F(t)), \quad 0 \le t < \infty .$$

Given the sample $X_1,\dots,X_n$, its empirical counterpart is

$$M_n(t) = \int_t^\infty (1-F_n(x))dx/(1-F_n(t)), \quad 0 \le t \le X_{n,n},$$

and the corresponding r_n process is

$$z_n(t) = n^{\frac{1}{2}}(M_n(t)-M_F(t))$$

(2.1)

$$= \{-\int_t^{\infty} n^{\frac{1}{2}}(F_n(x)-F(x))dx+M_F(t)n^{\frac{1}{2}}(F_n(t)-F(t))\}/(1-F_n(t)).$$

Theorem 4.1 of Cs-Cs-H (1986) concludes that there exists a sequence of Brownian bridges $B_n(y)$, $0 \leq y \leq 1$, such that for the sequence of identically distributed Gaussian processes

$$Z_F^{(n)}(t) = \{-\int_t^{\infty} B_n(F(x))dx + M_F(t)B_n(F(t))\}/(1-F(t))$$

we have the following: If $EX^2 < \infty$ and $T < \tau_F = \inf\{t : F(t) = 1\} \leq \infty$, then

$$\sup_{0\leq t\leq T} |z_n(t)-Z_F^{(n)}(t)| = o_P(1), \tag{2.2}$$

if $EX^2 < \infty$, then for $v_n(t) = (1-F_n(t))z_n(t)$, $0 \leq t < \infty$,

$$\sup_{0<t<\infty} |v_n(t)-(1-F(t))Z_F^{(n)}(t)| = o_P(1) \tag{2.3}$$

and, moreover, if $EX < \infty$ and $T < \tau_F$, then

$$\sup_{0\leq t\leq T} |M_n(t)-M_F(t)| = o(1) \quad \text{a.s.} \tag{2.4}$$

Here and in what follows all convergence and order relations are meant as $n \to \infty$ if not specified otherwise. The statements in (2.2), (2.3) and (2.4) are improvements over results of Yang (1978) and Hall and Wellner (1979).

Now we introduce the bootstrapped empirical process. Let m be a resampling size. Given $X_1,\ldots,X_n$, let $\tilde{X}_1,\ldots,\tilde{X}_m$ be conditionally independent rv's with common distribution function F_n. Let $\tilde{F}_{m,n}$ be the bootstrapped empirical distribution function, i.e., $\tilde{F}_{m,n}(t) =$

$m^{-1}\#\{k: 1 \le k \le m, \tilde{X}_k \le t\}$ given F_n fixed by $X_1,\ldots,X_n$. The bootstrapped empirical process is then

(2.5) $$m^{\frac{1}{2}}\{\tilde{F}_{m,n}(t)-F_n(t)\}, \quad -\infty < t < \infty.$$

Accordingly, the bootstrapped empirical mean residual life processes are

$$\tilde{z}_{m,n}(t) = m^{\frac{1}{2}}\{\tilde{M}_{m,n}(t)-M_n(t)\} \quad \text{and} \quad \tilde{v}_{m,n}(t) = (1-\tilde{F}_{m,n}(t))\tilde{z}_{m,n}(t),$$

where

$$\tilde{M}_{m,n}(t) = \int_t^\infty (1-\tilde{F}_{m,n}(x))dx/(1-\tilde{F}_{m,n}(t)).$$

(Wherever a denominator is zero, we define the corresponding process to be zero.) Our first theorem contains the bootstrapped versions of the results in (2.2) and (2.3).

THEOREM 1. *Suppose that* $EX^2 < \infty$ *and let* $m = m(n)$ *be a sequence of positive integers such that for two positive constants* $C_1 < C_2$

(2.6) $$C_1 m \le n \le C_2 m, \quad n=1,2,\ldots$$

Then we can define a sequence of Brownian bridges $\{\tilde{B}_m(y): 0 \le y \le 1\}_{m=1}^\infty$, *independent of the sequence* $\{X_n\}_{n=1}^\infty$, *such that for the identically distributed Gaussian processes*

$$\tilde{Z}_F^{(m)}(t) = \{-\int_t^\infty \tilde{B}_m(F(x))dx + M_F(t)\tilde{B}_m(F(t)) \ /(1-F(t))$$

we have

(2.7) $$\sup_{0\le t\le T} |\tilde{z}_{m,n}(t) - \tilde{Z}_F^{(m)}(t)| = o_P(1)$$

whenever $T < \tau_F$, *and*

$$\sup_{0 \leq t < \infty} |\tilde{v}_{m,n}(t) - (1-F(t))\tilde{Z}_F^{(m)}(t)| = o_P(1). \tag{2.8}$$

2.2. In the present subsection we assume that the lower endpoint of the support of F is the origin, i.e., $Q(0) = 0$, where Q is the quantile function of F given in (1.1). The second example of a statistical function R_F is the total time on test (ttt) function

$$H_F^{-1}(y) = \int_0^{Q(u)} (1-F(t))dt, \quad 0 \leq u < 1, \; H_F^{-1}(1) = \mu,$$

where we assume that $\mu = EX < \infty$ and that Q is continuous on [0,1). Given the sample $X_1,\ldots,X_n$, the empirical ttt function is

$$H_n^{-1}(u) = \int_0^{Q_n(u)} (1-F_n(t)dt$$

$$= n^{-1} \sum_{i=1}^{[[nu]]} (n+1-i)(X_{i,n}-X_{i-1,n}), \quad 0 \leq u < 1,$$

where for a nonnegative number x we denote by $[[x]]$ the smallest positive integer $\geq x$ (cf. (6.1) in Cs-Cs-H (1986) . Here $H_n^{-1}(1)$ is the sample mean $\bar{X}_n$. Sometimes scaled versions of these functions are used, i.e., the scaled ttt transform and the scaled empirical ttt transform:

$$D_F^{-1}(u) = H_F^{-1}(u)/\mu, \quad D_n^{-1}(u) = H_n^{-1}(u)/\bar{X}_n, \quad 0 \leq u \leq 1.$$

Let us introduce the empirical ttt and scaled ttt processes

$$t_n(u) = n^{\frac{1}{2}}(H_n^{-1}(u)-H_F^{-1}(u)) \quad \text{and} \quad s_n(u) = n^{\frac{1}{2}}(D_n^{-1}(u)-D_F^{-1}(u)), \; 0 \leq u \leq 1.$$

For a bibliography of literature on these processes we refer to Cs-Cs-H (1986). Using the same Brownian bridges B_n as in $Z_F^{(n)}$, we consider the sequence of identically distributed Gaussian processes, defined by continuity at 0 and 1,

$$(2.9)\qquad T_F^{(n)}(u) = -\int_0^u B_n(s)\,dQ(s) - \frac{1-u}{f(Q(u))} B_n(u), \quad 0 \le u \le 1,$$

where f is the density function of F, and consider also the identically distributed Gaussian processes

$$(2.10)\quad S_F^{(n)}(u) = \mu^{-1} T_F^{(n)}(u) - \mu^{-2} H_F^{-1}(u)\, T_F^{(n)}(u), \quad 0 \le u \le 1.$$

Let $\mathcal{Q}$ denote the class of positive functions q defined on $(0,1)$ such that for any $q \in \mathcal{Q}$ there exists a $0 < \delta < 1/2$ and an $\varepsilon > 0$ such that $q(s) \ge \varepsilon$ for all $\delta < s < 1-\delta$ and both $q(s)$ and $q(1-s)$ are nondecreasing on $(0,\delta]$. A function $q \in \mathcal{Q}$ will be called a Chibisov-O'Reilly function if and only if

$$(2.11)\quad \int_0^{1/2} s^{-1} e^{-cq^2(s)/s}\,ds < \infty \quad \text{and} \quad \int_0^{1/2} s^{-1} e^{-cq^2(1-s)/s}\,ds < \infty$$

for all $c > 0$. By Theorems 6.2 and 7.2 of Cs-Cs-H (1986) we know that if the density function $f = F'$ is continuous and positive on the open support of F, $EX^2 < \infty$, and there exists a Chibisov-O'Reilly function q such that

$$(2.12)\qquad \sup_{0<u<1} \frac{q(u)(1-u)}{f(Q(u))} < \infty,$$

then

$$(2.13)\quad \sup_{0\le u\le 1} |t_n(u)-T_F^{(n)}(u)| = o_P(1) \quad\text{and}\quad \sup_{0\le u\le 1} |s_n(u)-S_F^{(n)}(u)| = o_P(1).$$

(In Cs-Cs-H (1986) it is assumed that q is continuous. It follows from results in Cs-Cs-H (1986) that this assumption can be dropped.)

Let

$$(2.14)\quad \tilde{Q}_{m,n}(s) = \inf\{t: \tilde{F}_{m,n}(t) \ge s\},\quad 0 < s \le 1,\ \tilde{Q}_{m,n}(0) = \tilde{Q}_{m,n}(0+),$$

be the bootstrapped empirical quantile function, where $F_{m,n}$ is as in (2.5). Then the bootstrapped empirical ttt and scaled ttt processes are

$$\tilde{t}_{m,n}(u) = m^{\frac{1}{2}}\{\tilde{H}^{-1}_{m,n}(u)-H^{-1}_n(u)\} \text{ and } \tilde{s}_{m,n}(u) = m^{\frac{1}{2}}\{\tilde{D}^{-1}_{m,n}(u)-D^{-1}_n(u)\},$$

where

$$\tilde{H}^{-1}_{m,n}(u) = \int_0^{\tilde{Q}_{m,n}(u)} (1 - \tilde{F}_{m,n}(t))dt$$

$$= m^{-1} \sum_{i=1}^{[[mu]]} (m+1-i)(\tilde{X}_{i,m}-\tilde{X}_{i-1,m}),$$

where $\tilde{X}_{1,m} \le \dots \le \tilde{X}_{m,m}$ are the order statistics of $\tilde{X}_1,\dots,\tilde{X}_m$ and the last equation holds uniformly almost surely, and

$$\tilde{D}^{-1}_{m,n}(u) = \tilde{H}^{-1}_{m,n}(u)/(\frac{1}{m}\sum_{i=1}^{m}\tilde{X}_i),\quad 0 \le u \le 1.$$

Let $\tilde{T}_F^{(m)}(u)$, $0 < u < 1$, denote the Gaussian process obtained upon replacing B_n in (2.9) by $\tilde{B}_m$ of Theorem 1 and let $\tilde{S}_F^{(m)}$ be the Gaussian process obtained upon replacing $T_F^{(n)}$ in (2.10) by $\tilde{T}_F^{(m)}$. The bootstrapped versions of the results in (2.13) are the following.

THEOREM 2. *Suppose that the density function* $f = F'$ *is continuous and positive on the open support of* F , $EX^2 < \infty$, *and that there exists a Chibisov-O'Reilly function* q *such that (2.12) holds. If* $m = m(n)$ *is a sequence of positive integers that satisfies condition (2.6), then*

$$\sup_{0 \le u \le 1} |\tilde{t}_{m,n}(u) - \tilde{T}_F^{(m)}(u)| = o_P(1) \tag{2.15}$$

and

$$\sup_{0 \le u \le 1} |\tilde{s}_{m,n}(u) - \tilde{S}_F^{(m)}(u)| = o_P(1). \tag{2.16}$$

2.3. We again assume that Q is continuous on $[0,1)$ and that $\mu = EX < \infty$, however $Q(0) \geq 0$ can be positive. Our third example of the statistical function R_F is the Lorenz curve

$$L_F(u) = \frac{1}{\mu} \int_0^{Q(u)} x dF(x) = \frac{1}{\mu} \int_0^u Q(s) ds, \quad 0 \le u < 1, \; L_F(1) = 1.$$

Given the sample $X_1, \ldots, X_n$, we define the empirical Lorenz curve as

$$L_n(u) = \frac{1}{\bar{X}_n} \int_0^{Q_n(u)} x dF_n(x)$$

$$= \frac{1}{\bar{X}_n} \frac{1}{n} \sum_{i=1}^{[[nu]]} X_{i,n}, \quad 0 \le u \le 1.$$

The empirical Lorenz process (cf. Goldie (1977) and Cs-Cs-H (1986) for bibliographies on applications) is then

$$\ell_n(u) = n^{\frac{1}{2}}(L_n(u) - L_F(u)), \quad 0 \le u \le 1.$$

Now we consider the following sequence of identically distributed

Gaussian processes

$$(2.17)\qquad \Lambda_F^{(n)}(u) = \frac{1}{\mu}\{-\int_0^u B_n(s)dQ(s) + L_F(u)\int_0^1 B_n(s)dQ(s)\},\quad 0 \le u \le 1.$$

Theorem 11.2 of Cs-Cs-H (1986) concludes that if $EX^2 < \infty$, then

$$(2.18)\qquad \sup_{0\le u\le 1} |\ell_n(u) - \Lambda_F^{(n)}(u)| = o_P(1).$$

Let again $\tilde{F}_{m,n}$ be as in (2.5) and $\tilde{Q}_{m,n}$ is as in (2.14). The bootstrapped Lorenz process is then

$$\tilde{\ell}_{m,n}(u) = m^{\frac{1}{2}}\{\tilde{L}_{m,n}(u) - L_n(u)\},\quad 0 \le u \le 1,$$

where

$$\tilde{L}_{m,n}(u) = \int_0^{\tilde{Q}_{m,n}(u)} x d\tilde{F}_{m,n}(x)/(\frac{1}{m}\sum_{i=1}^m \tilde{X}_i)$$

$$= (\frac{1}{m}\sum_{i=1}^{[[mu]]} \tilde{X}_{i,m})/(\frac{1}{m}\sum_{i=1}^m \tilde{X}_i),\quad 0 \le u \le 1.$$

Also, let $\tilde{\Lambda}_F^{(m)}(u)$, $0 \le u \le 1$, denote the Gaussian process obtained upon replacing B_n in (2.17) by $\tilde{B}_m$ of Theorems 1 and 2. The bootstrapped version of (2.18) is the following.

THEOREM 3. *If* $EX^2 < \infty$ *and* $m = m(n)$ *is a sequence of positive integers such that condition (2.6) holds, then*

$$(2.19)\qquad \sup_{0\le u\le 1} |\tilde{\ell}_{m,n}(u) - \tilde{\Lambda}_F^{(m)}(u)| = o_P(1).$$

In order to form asymptotically correct bootstrapped confidence bands as described in the introduction for the above statistical functions M_F, H_F^{-1}, D_F^{-1} and L_F, one ingredient is still missing. We need to know that the distribution functions of the absolute suprema of the corresponding Gaussian processes are continuous. This fact is an immediate consequence of a result if Tsirel'son (1975).

LEMMA 1. *Let* G *be a separable Gaussian process defined on a closed interval* $[a,b]$. *Assume that* G *is almost surely bounded on* $[a,b]$ *and* $\operatorname{var} G(s) > 0$ *for some* $s \in (a,b)$. *Then the distribution function*

$$G(y) = \Pr\{\sup_{a\leq s\leq b} |G(s)| \leq y\} \quad \text{for} \quad 0 < y < \infty.$$

is continuous on $(0,\infty)$.

3. A simulation study and an example. Cs-Cs-H (1986) proved that if F is the exponential distribution function then the scaled ttt process $\{s_n(u): 0 \leq u \leq 1\}$ converges weakly to a Brownian bridge process. Thus in this case

$$\lim_{n\to\infty} P\{\sup_{0\leq u\leq 1} |s_n(u)| \leq x\} = K(x)$$

$$= 1 - \sum_{k\neq 0} (-1)^{k+1} \exp(-2k^2x^2) = (2\pi)^{\frac{1}{2}}x^{-1} \sum_{k=1}^{\infty} \exp(-\pi^2(2k-1)^2/(8x^2)),$$

and we have also

$$K_{N,m,n}(x) = \frac{1}{N}\#\{1\leq i\leq N: m^{\frac{1}{2}} \sup_{0\leq u\leq 1} |\tilde{D}_{m,n}^{-1(i)}(u) - D_n^{-1}(u)| \leq x\} \longrightarrow K(x) \text{ a.s.},$$

as $N,m,n \to \infty$. Simulations were carried out for exponential data with sample sizes of $n = 10, 20, 50, 100$ and 200, each repeated 100 times. Bootstrap samples of size $m = n$ were drawn $N = 1000$ times and the deviation of critical values $|K^{-1}(\alpha) - K^{-1}_{N,m,n}(\alpha)|$ is reported in Table 1. We note that the difference between the bands using the limit distribution and the bootstrap procedure is about $0.2n^{-\frac{1}{2}}$ for these moderate sample sizes.

As an example we examined 107 failure times for right rear brakes on D9G-66A Caterpillar tractors. These data are available in Barlow and Campo (1975) and Doksum and Yandell (1984), where total time on test and other tests of exponentiality are investigated. Here we present asymptotically 90% bootstrap confidence bands for the ttt transform H_F^{-1}, the scaled ttt transform D_F^{-1} and the Lorenz curve L_F of these data (Figure 1). Bootstrap samples of size $m = 107$ were drawn $N = 1000$ times.

The bands on Figure 1 have constant width. In some statistical applications it may be useful to have the width of the bands depend on the variance of the underlying process at time t. It is easy to get this type of bands with a minor modification of (1.10). Returning to (1.3), let

$$\sigma^2(t) = E(G_F(t) - EG_F(t))^2$$

and assume that

$$\inf_{t \in I} \sigma(t) > 0. \tag{3.1}$$

If we can find an estimator $\sigma_n(t)$ of $\sigma(t)$, which satisfies

$$(3.2) \qquad \sup_{t \in I} |\sigma_n(t) - \sigma(t)| = o_P(1),$$

then

$$(3.3) \qquad P\{R_n(t) - \tilde{c}_{N,m,n} n^{-\frac{1}{2}} \sigma_n(t) \leq R_F(t) \leq R_n(t) + \tilde{c}_{N,m,n} n^{-\frac{1}{2}} \sigma_n(t) : t \in I\} \to 1-\alpha$$

as $N,m,n \to \infty$, where

$$\tilde{G}_{N,m,n}(x) = \frac{1}{N} \# \{1 \leq i \leq N : m^{\frac{1}{2}} \sup_{t \in I} |\tilde{R}^{(i)}_{m,n}(t) - R_n(t)|/\sigma_n(t) \leq x\}$$

and

$$\tilde{c}_{N,m,n} = \inf \{x : \tilde{G}_{N,m,n}(x) \geq 1-\alpha\}.$$

Cs-Cs-H (1986) showed that the limit processes of the empirical ttt, scaled ttt and Lorenz processes satisfy (3.1) when $I = [a,b]$, $0 < a \leq b < 1$. They also gave estimators for the variances of the limit processes for which (3.2) holds. Figure 2 shows confidence bands of the form (3.3) for H_F^{-1}, D_F^{-1} and L_F on $[0.1, 0.9]$ using the rear breaks data. The bootstrap samples of size $m = 107$ were drawn $N = 1000$ times again.

4. Technical tools. In the previous sections we assumed that our probability space (Ω, A, P) is so rich that it accommodates all the rv's and processes introduced up to now or later on. This can be done by results of M. Csörgő and Révész (1981), Berkes and Philipp (1979) and De Acosta (1982), without loss of generality. Let $U_1, U_2, \ldots$ be

a sequence of independent uniform-(0,1) rv's and define the uniform empirical and quantile processes

$$\alpha_n(s) = n^{\frac{1}{2}}(G_n(s)-s) \quad \text{and} \quad \beta_n(s) = n^{\frac{1}{2}}(s-U_n(s)), \quad 0 \le s \le 1,$$

where $G_n(s) = n^{-1}\#\{k : 1 \le k \le n,\ U_k \le s\}$ and, with $U_{1,n} \le \dots \le U_{n,n}$ denoting the order statistics of $U_1,\dots,U_n$,

$$U_n(s) = \begin{cases} U_{k,n}, & (k-1)/n < s \le k/n, \quad k = 1,\dots,n, \\ U_{1,n}, & s = 0 . \end{cases}$$

We can and will assume that $X_i = Q(U_i)$, $1 \le i \le n$, holds. Cs-Cs-H-M (1986) constructed a sequence of Brownian bridges $\{B_n(s) : 0 \le s \le 1\}$ such that

$$(4.1) \quad \sup_{0<s<1} \frac{|\alpha_n(s)-B_n^*(s)|}{(s(1-s))^{1/2-\nu_1}} = O_P(n^{-\nu_1}) \text{ and } \sup_{\lambda/n\le s\le 1-\lambda/n} \frac{|\beta_n(s)-B_n(s)|}{(s(1-s))^{1/2-\nu_2}} = O_P(n^{-\nu_2})$$

for all $0 < \lambda < \infty$ and $0 \le \nu_1 < 1/4$, $0 \le \nu_2 < 1/2$, where $B_n^*(s) = B_n(s)$ if $1/n \le s \le 1-1/n$ and zero otherwise.

Let $\xi_1, \xi_2, \dots$ be a sequence of independent uniform (0,1) rv's, independent from $\{U_i\}_{i=1}^{\infty} \cup \{B_i(s) : 0 \le s \le 1\}_{i=1}^{\infty}$. We define again the corresponding empirical and quantile processes

$$e_m(s) = m^{\frac{1}{2}}(E_m(s)-s) \quad \text{and} \quad k_m(s) = m^{\frac{1}{2}}(s-\xi_m(s)), \quad 0 \le s \le 1,$$

where $E_m(s) = m^{-1}\#\{k : 1 \le k \le m,\ \xi_k \le s\}$, and, with $\xi_{1,m} \le \dots \le \xi_{m,m}$ denoting the order statistics of $\xi_1,\dots,\xi_m$,

$$\xi_m(s) = \begin{cases} \xi_{k,m}, & (k-1)/m < s \le k/m, \quad k = 1,\dots,m, \\ \xi_{1,m}, & s = 0. \end{cases}$$

By Cs-Cs-H-M (1986) we can define another sequence of Brownian bridges $\{\hat{B}_m(s): 0 \le s \le 1\}_{m=1}^{\infty}$, such that $\{U_i\}_{i=1}^{\infty} \cup \{B_i(s): 0 \le s \le 1\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty} \cup \{\hat{B}_i(s): 0 \le s \le 1\}_{i=1}^{\infty}$ are independent and we also have, as $m \to \infty$,

$$(4.2) \quad \sup_{0<s<1} \frac{|e_m(s)-\hat{B}_m^*(s)|}{(s(1-s))^{\frac{1}{2}-\nu_1}} = O_P(m^{-\nu_1}) \text{ and } \sup_{\lambda/m\le s\le 1-\lambda/m} \frac{|k_m(s)-\hat{B}_m(s)|}{(s(1-s))^{\frac{1}{2}-\nu_2}} = O_P(m^{-\nu_2})$$

for all $0 < \lambda < \infty$ and $0 \le \nu_1 < 1/4$, $0 \le \nu_2 < 1/2$, where $\hat{B}_m^*(s) = \hat{B}_m(s)$ if $1/m \le s \le 1-1/m$ and zero otherwise.

For convenient reference later on we list a number of facts on the linearity of the uniform empirical distribution and quantile functions:

$$(4.3) \quad \sup_{0<s<1} G_n(s)/s + \sup_{0<s<1} (1-G_n(s))/(1-s) = O_P(1) ,$$

$$(4.4) \quad \sup_{U_{1,n}\le s\le 1} s/G_n(s) + \sup_{0<s<U_{n,n}} (1-s)/(1-G_n(s)) = O_P(1),$$

$$(4.5) \quad \sup_{0<s<1} s/U_n(s) + \sup_{0<s<1} (1-s)/(1-U_n(s)) = O_P(1),$$

and for any $0 < \rho < \infty$

$$(4.6) \quad \sup_{\rho/n\le s<1} U_n(s)/s + \sup_{0\le s\le 1-\rho/n} (1-U_n(s))/(1-s) = O_P(1).$$

All these follow from Remark 1 of Wellner (1978). Of course, all these statements hold also for the empirical distribution and quantile functions $E_m(\cdot)$ and $\xi_m(\cdot)$ of the $\{\xi_i\}$ sequence.

Introduce

$$(4.7) \qquad G_{m,n}(s) = E_m(G_n(s)) \quad\text{and}\quad U_{m,n}(s) = U_n(\xi_m(s)),\ 0 \le s \le 1,$$

and the quantile function

$$Q_n(s) = \inf\{x\colon F_n(x) \ge s\},\ 0 < s \le 1,\ Q(0) = Q(0+)$$

of the original sample. It is easy to show that

$$(4.8) \qquad \begin{aligned} &\{(\tilde{F}_{m,n}(x),\tilde{Q}_{m,n}(s),F_n(y),Q_n(t))\colon -\infty < x,\ y < \infty,\ 0 \le s,\ t \le 1\} \\ &\overset{\mathcal{D}}{=} \{(G_{m,n}(F(x)),Q(U_{m,n}(s)),G_n(F(y)),Q(U_n(t)))\colon -\infty<x,y<\infty\ ,\ 0\le s,t\le 1\}. \end{aligned}$$

(This distributional equivalence was used implicitly by Bickel and Freedman (1981) and by Shorack (1982).) For this reason we shall sometimes refer to the processes

$$(4.9) \quad \alpha_{m,n}(s) = m^{\frac{1}{2}}(G_{m,n}(s)-G_n(s)) \quad\text{and}\quad \beta_{m,n}(s) = m^{\frac{1}{2}}(U_n(s)-U_{m,n}(s)),\ 0\le s\le 1,$$

as the bootstrapped uniform empirical and quantile processes, respectively. For these processes we need the "bootstrapped" versions of (4.1) and (4.2).

PROPOSITION 1. *For any sequence* $m = m(n) \to \infty$ *of positive integers and for each* $0 \le \nu < 1/4$

$$(4.10) \qquad \sup_{U_{1,n}\le s\le U_{n,n}} |\alpha_{m,n}(s)-\hat{B}^*_m(s)|/(s(1-s))^{\frac{1}{2}-\nu} = O_P((m \wedge n)^{-\nu}),$$

and whenever $m = m(n)$ *satisfies condition (2.6), for all* $0 < \lambda < \infty$ *and* $0 \le \nu < 1/4$

$$(4.11)\qquad \sup_{\lambda/m \le s \le 1-\lambda/m} |\beta_{m,n}(s) - \hat{B}_m(s)|/(s(1-s))^{\frac{1}{2}-\nu} = O_P(m^{-\nu}).$$

Proof. First we consider (4.10). Choose any $0 \le \nu < 1/4$. Since (4.3) implies that

$$\sup_{0<s<1} G_n(s)(1-G_n(s))/(s(1-s)) = O_P(1),$$

it is sufficient to prove that

$$(4.12)\qquad S_{m,n}^{(\nu)} := \sup_{U_{1,n} \le s < U_{n,n}} \Delta_{m,n}^{(\nu)}(s) = O_P((n \wedge m)^{-\nu}),$$

where

$$\Delta_{m,n}^{(\nu)}(s) = |\alpha_{m,n}(s) - \hat{B}_m^*(s)|/(G_n(s)(1-G_n(s)))^{\frac{1}{2}-\nu}.$$

Observe that

$$S_{m,n}^{(\nu)} \le \sup_{U_{1,n} \vee \frac{1}{m} \le s < U_{n,n} \wedge (1-\frac{1}{m})} \Delta_{m,n}^{(\nu)}(s) + \sup_{U_{1,n} \le s < U_{1,n} \vee \frac{1}{m}} \Delta_{m,n}^{(\nu)}(s)$$

$$+ \sup_{U_{n,n} \wedge (1-\frac{1}{m}) \le s < U_{n,n}} \Delta_{m,n}^{(\nu)}$$

$$=: S_{m,n,1}^{(\nu)} + S_{m,n,2}^{(\nu)} + S_{m,n,3}^{(\nu)},$$

where $S_{m,n,2}^{(\nu)}$ is defined to be zero if $U_{1,n} \vee (1/m) = U_{1,n}$ and $S_{m,n,3}^{(\nu)}$ is defined to be zero if $U_{n,n} \wedge (1 - 1/m) = U_{n,n}$.

Notice that

$$S^{\nu}_{m,n,1} \le \sup_{U_{1,n} \vee \frac{1}{m} < s < U_{n,n} \wedge (1-\frac{1}{m})} |\alpha_{m,n}(s) - B^*_m(G_n(s))| / (G_n(s)(1-G_n(s)))^{\frac{1}{2}-\nu}$$

$$+ \sup_{U_{1,n} \vee \frac{1}{m} < s < U_{n,n} \wedge (1-\frac{1}{m})} |\hat{B}_m(G_n(s)) - \hat{B}_m(s)| / (G_n(s)(1-G_n(s)))^{\frac{1}{2}-\nu}$$

$$\le \sup_{0<s<1} |e_m(s) - \hat{B}^*_m(s)| / (s(1-s))^{\frac{1}{2}-\nu}$$

$$+ \sup_{U_{1,n} < s < U_{n,n}} |\hat{B}_m(G_n(s)) - \hat{B}_m(s)| / (G_n(s)(1-G_n(s)))^{\frac{1}{2}-\nu} .$$

By (4.2) the first term on the right side is $O_P(m^{-\nu})$, while the second term is

$$\le 2 \max_{1 \le i \le n-1} \sup_{U_{i,n} \le s < U_{i+1,n}} |\hat{B}_m(i/n) - \hat{B}_m(s)| / (i/n)^{\frac{1}{2}-\nu}$$

$$+ 2 \max_{1 \le i \le n-1} \sup_{U_{i,n} \le s < U_{i+1,n}} |\hat{B}_m(i/n) - \hat{B}_m(s)| / (1-i/n)^{\frac{1}{2}-\nu} .$$

In the proof of Theorem 2.2 of Cs-Cs-H-M (1986) it was shown that these last two terms are $O_P(n^{-\nu})$. Hence we have

$$S^{(\nu)}_{m,n,1} = O_P((m \wedge n)^{-\nu}). \tag{4.13}$$

Next we consider $S^{(\nu)}_{m,n,2}$. Note that since $\hat{B}^*_m(s) = 0$ for $0 \le s < 1/m$,

$$S^{(\nu)}_{m,n,2} = \sup_{U_{1,n} \le s < U_{1,n} \vee \frac{1}{m}} |\alpha_{m,n}(s)| / (G_n(s)(1-G_n(s)))^{\frac{1}{2}-\nu}$$

$$\le \sup_{0<s<G_n(1/m)} |e_m(s)| / (s(1-s))^{\frac{1}{2}-\nu} .$$

Choose any $\rho > 1$. Notice that whenever $G_n(1/m) \leq \rho/m$, this last expression is

$$\leq \sup_{0<s<\rho/m} |e_m(s)|/(s(1-s))^{\frac{1}{2}-\nu}$$

$$\leq m^{-\nu} \sup_{0<s<\rho/m} (ms)^{\frac{1}{2}+\nu}/(1-s)^{\frac{1}{2}-\nu}$$

$$+ m^{-\nu}\{mE_m(\rho/m)\}/\{(m\xi_{1,m})^{\frac{1}{2}-\nu}(1-\rho/m)^{\frac{1}{2}-\nu}\}$$

$$\leq 2m^{-\nu} + O_P(m^{-\nu})mE_m(\rho/m)$$

for large enough m and by (4.5). Since $E(mE_m(\rho/m)) = \rho$, the last two terms are $O_P(m^{-\nu})$. On the other hand, by Markov's inequality

$$P\{G_n(1/m) \leq \rho/m\} \geq 1-1/\rho$$

for all $\rho > 1$ and $m \geq 1$, and therefore an elementary argument now establishes that

$$S^{(\nu)}_{m,n,2} = O_P(m^{-\nu}).$$

An analogous proof shows that we also have

$$S^{(\nu)}_{m,n,3} = O_P(m^{-\nu}).$$

The last two relations and (4.13) imply (4.12) and hence the first statement of the proposition.

To prove (4.11), first we observe that

$$|\beta_{m,n}(s)-\hat{B}_m(s)| \leq |k_m(s)-\hat{B}_m(s)| + (\frac{m}{n})^{\frac{1}{2}}|\beta_n(\xi_m(s))-B_n(\xi_m(s))|$$

$$+ (\frac{m}{n})^{\frac{1}{2}}|B_n(\xi_n(s))-B_n(s)| + (\frac{m}{n})^{\frac{1}{2}}|B_n(s)-\beta_n(s)|$$

$$=: \nabla_m^{(1)}(s) +\ldots+ \nabla_m^{(4)}(s).$$

Choose any $0 \leq \nu < 1/4$. By (4.2) we have

$$\sup_{\lambda/m\leq s\leq 1-\lambda/m} \nabla_m^{(1)}(s)/(s(1-s))^{\frac{1}{2}-\nu} = O_P(m^{-\nu}),$$

and by (4.1) along with assumption (2.6) we obtain

$$\sup_{\lambda/m\leq s\leq 1-\lambda/m} \nabla_m^{(4)}(s)/(s(1-s))^{\frac{1}{2}-\nu} = O_P(m^{-\nu}).$$

Choose any $1 < \rho < \infty$ and set

$$A_m^{(\lambda)}(\rho) = \{s/\rho<\xi_m(s) \text{ and } \xi_m(s) \leq 1-(1-s)/\rho \text{ for } \lambda/m \leq s \leq 1-\lambda/m\}.$$

Notice that on the event $A_m^{(\lambda)}(\rho)$

$$\sup_{\lambda/m\leq s\leq 1-\lambda/m} \nabla_m^{(2)}(s)/(s(1-s))^{\frac{1}{2}-\nu} \leq \rho^{1-2\nu}(\frac{m}{n})^{\frac{1}{2}} \sup_{\lambda/(\rho m)\leq t\leq 1-\lambda/(\rho m)} |\beta_n(t)-B_n(t)|/(t(1-t))^{\frac{1}{2}-\nu}$$

which by (4.1) and (2.6) is $O_P(n^{-\nu}) = O_P(m^{-\nu})$. But (4.5) as applied to $\xi_m(\cdot)$ implies that

$$\lim_{\rho\to\infty} \liminf_{m\to\infty} P\{A_m^{(\lambda)}(\rho)\} = 1.$$

Hence

$$\sup_{\lambda/m \leq s < 1-\lambda/m} \nabla_m^{(2)}(s)/(s(1-s))^{\frac{1}{2}-\nu} = O_P(m^{-\nu}).$$

Finally, to establish that

$$\sup_{\lambda/m \leq s \leq 1-\lambda/m} \nabla_m^{(3)}(s)/(s(1-s))^{\frac{1}{2}-\nu} = O_P(m^{-\nu}),$$

one requires a routine, though very lengthy, modification of the corresponding part of the proof of Theorem 2.2 of Cs-Cs-H-M (1986). For the sake of brevity these details are omitted.

We note that if (2.6) holds, then we can replace B_m^* by B_m in (3.10).

Next we state the Chibisov-O'Reilly theorem for the bootstrapped uniform empirical and quantile processes. These results can be deduced from Proposition 1 in exactly the same way as the corresponding results for the oridinary processes were derived from (4.1) and (4.2) (cf. the first proof of Theorem 4.2.1 and Corollary 4.3.1 in Cs-Cs-H-M (1986)). Recall the definition of the class $\mathcal{Q}$ from Section 2 (cf. (2.11)).

PROPOSITION 2. *Let* $m = m(n)$ *be any sequence of positive integers converging to infinity. For any* $q \in \mathcal{Q}$

$$\sup_{0<s<1} |\alpha_{m,n}(s) - \hat{B}_m(s)|/q(s) = o_P(1)$$

if and only if q *is a Chibisov-O'Reilly function, i.e., if and only if the integrals in (2.11) converge for all* $c > 0$. *Also, if* $q \in \mathcal{Q}$ *is a Chibisov-O'Reilly function and if condition (2.6) is satisfied then for any* $0 < \lambda < \infty$

$$\sup_{\lambda/m \leq s \leq 1-\lambda/m} |\beta_{m,n}(s)-B_m(s)|/q(s) = o_P(1).$$

We note that the second statement is in fact true under the weaker assumption that $q \in \mathcal{Q}$ is such that both $q(s)/s^{\frac{1}{2}}$ and $q(1-s)/s^{\frac{1}{2}}$ converge to infinity as $s \downarrow 0$.

Our final technical proposition gives the analogues of the linearity statements in (4.3)-(4.6) for $G_{m,n}$ and $U_{m,n}$ in (4.7).

PROPOSITION 3. *Let* $m = m(n)$ *denote any sequence of positive integers converging to infinity. Then*

$$\sup_{0<s<1} G_{m,n}(s)/s + \sup_{0<s<1} (1-G_{m,n}(s))/(1-s) = O_P(1), \tag{4.14}$$

$$\sup_{U_{m,n}(0)\leq s<1} s/G_{m,n}(s) + \sup_{0<s<U_{m,n}(1)} (1-s)/(1-G_{m,n}(s)) = O_P(1), \tag{4.15}$$

$$\sup_{0<s<1} s/U_{m,n}(s) + \sup_{0<s<1} (1-s)/(1-U_{m,n}(s)) = O_P(1), \tag{4.16}$$

and whenever there exists a constant $0 < C < \infty$ *such that* $m/n \leq C$ *for all* $n \geq 1$, *then for any* $0 < \lambda < \infty$

$$\sup_{\lambda/m\leq s<1} U_{m,n}(s)/s + \sup_{0<s\leq 1-\lambda/m} (1-U_{m,n}(s))/(1-s) = O_P(1). \tag{4.17}$$

Proof. Combining (4.7) and (4.3)-(4.5) we immediately obtain the results.

5. Proofs of Theorems 1,2,3.

Recall that in Section 2 we assumed that our underlying rv X is nonnegative. This will be assumed in the present section without further mention. Before starting the proofs we introduce further processes. Let

$$\hat{z}_{m,n}(t) = \{-\int_t^{\infty} \alpha_{m,n}(F(x))dx + \alpha_{m,n}(F(t))M_n(t)\}/(1-G_{m,n}(F(t)))$$

$$= \{-\int_{F(t)}^{1} \alpha_{m,n}(s)dQ(s) + \alpha_{m,n}(F(t))M_n(Q(F(t)))\}/(1-G_{m,n}(F(t))),$$

where the second inequality is obtained by the change of variables $F(x) = s$. Also we define

$$\hat{t}_{m,n}(s) = m^{\frac{1}{2}}\{\int_0^{Q(U_{m,n}(s))} (1-G_{m,n}(F(x)))dx - \int_0^{Q(U_n(s))} (1-G_n(F(x)))dx$$

$$= m^{\frac{1}{2}}\{\int_0^{U_{m,n}(s)} (1-G_{m,n}(s))dQ(s) - \int_0^{U_n(s)} (1-G_n(s))dQ(s)\}$$

$$=: m^{\frac{1}{2}}\{\hat{H}_{m,n}^{-1}(s) - H_n^{-1}(s)\}$$

again by the change of variables $F(x) = s$, along with the assumption that here $Q(0) = 0$ and both F and Q are continuous. Finally, let

$$\hat{\ell}_{m,n}(s) = m^{\frac{1}{2}}\{(\int_0^{Q(U_{m,n}(s))} xdG_{m,n}(F(x))/\int_0^{\infty} xdG_{m,n}(F(x)))$$

$$- (\int_0^{Q(U_n(s))} xdG_n(F(x))/\int_0^{\infty} xdG_n(F(x)))\}$$

$$= m^{\frac{1}{2}}\{\int_0^{U_{m,n}(s)} Q(t)dG_{m,n}(t)/\int_0^{1} Q(t)dG_{m,n}(t)$$

$$- \int_0^{U_n(s)} Q(t)dG_n(t)/\int_0^{1} Q(t)dG_n(t)\}.$$

Using the representations for z_n, t_n and ℓ_n given in Cs-Cs-H (1986) and (4.8), one can easily check that

$$(5.1)\quad \{z_n(s),\ \tilde{z}_{m,n}(t)\ 0\leq s,t<\infty\} \overset{\mathcal{D}}{=} \{z_n(s),\ \hat{z}_{m,n}(t)\quad 0\leq s,t<\infty\}$$

for each $m,n \geq 1$, and also

$$(5.2)\quad \{z_n(s),\ t_n(x),\ \ell_n(y),\ \tilde{z}_{m,n}(t),\ \tilde{t}_{m,n}(u),\ \tilde{\ell}_{m,n}(v):\ 0\leq s,t<\infty,\ 0\leq x,y,u,v\leq 1\}$$

$$\overset{\mathcal{D}}{=} \{z_n(s), t_n(x), \ell_n(y), \hat{z}_{m,n}(t), \hat{t}_{m,n}(u), \hat{\ell}_{m,n}(v):\ 0\leq s,t<\infty,\ 0\leq x,y,u,v\leq 1\}$$

for each $m,n \geq 1$, if F and Q are continuous.

The following lemma in Cs-Cs-H (1986) is a basic ingredient in the proofs of (2.2), (2.3), (2.13) and (2.19). It is Lemma 3.2 there and for easy access for later use we quote it here.

LEMMA 2. *If* $EX^2 < \infty$, *then*

$$(5.3)\qquad \sup_{0\leq y\leq 1} \left|\int_0^y \{B_n(s)-\alpha_n(s)\}dQ(s)\right| = o_P(1).$$

We need the analogous result for the bootstrapped uniform empirical process with the corresponding sequence $\{\tilde{B}_m\}$ of Brownian bridges. In what follows C will denote a generic constant, not necessarily the same at each appearance.

LEMMA 3. *If* $EX^2 < \infty$ *and condition (2.6) is satisfied, then*

$$\Delta_m^{(1)} = \sup_{0\leq y\leq 1} \left|\int_0^y \{\hat{B}_m(s)-\alpha_{m,n}(s)\}dQ(s)\right| = o_P(1).$$

Proof. By simple manipulation

$$\Delta_m^{(1)} \leq \sup_{0\leq y\leq 1} \left|\int_0^y \{\hat{B}_m^*(s)-\alpha_{m,n}(s)\}dQ(s)\right|$$

$$+ \sup_{0\le y\le 1} \left|\int_0^y \{\hat{B}_m(s)-\hat{B}_m^*(s)\}dQ(s)\right|$$

$$\le \int_{U_{1,n}}^{U_{n,n}} |\hat{B}_m^*(s)-\alpha_{m,n}(s)|dQ(s)$$

(5.4)

$$+ \int_0^{U_{1,n}} |\hat{B}_m^*(s)|dQ(s) + \int_{U_{n,n}}^1 |\hat{B}_m^*(s)|dQ(s)$$

$$+ \sup_{0\le y\le 1/m} \left|\int_0^y \hat{B}_m(s)dQ(s)\right| + \sup_{1-1/m\le y\le 1} \left|\int_{1-1/m}^y \hat{B}_m(s)dQ(s)\right|$$

$$=: \Delta_{m,1}^{(1)} +\ldots+ \Delta_{m,5}^{(1)} .$$

In the proof of Lemma 4 (Lemma 3.2 in Cs-Cs-H (1986)) we have shown that the condition $EX^2 < \infty$ implies via the Birnbaum-Marshall inequality that

$$\Delta_{m,4}^{(1)} + \Delta_{m,5}^{(1)} = o_P(1). \tag{5.5}$$

For any $\lambda > 1$ consider the event

$$\Omega_n(\lambda) = \{\frac{1}{n\lambda} \le U_{1,n} \le \frac{1}{n}, \ 1-\frac{\lambda}{n} \le U_{n,n} \le 1-\frac{1}{n\lambda}\} .$$

On $\Omega_n(\lambda)$ we have

$$\Delta_{m,3}^{(1)} \le \int_{1-\lambda/n}^1 |\hat{B}_m^*(s)|dQ(s).$$

Using condition (2.6) we obtain

$$E\int_{1-\lambda/n}^1 |\hat{B}_m^*(s)|dQ(s) \le C\, n^{-\frac{1}{2}}\{Q(1-\lambda/n) + Q(1-C_1/n)\},$$

and this bound goes to zero since

(5.6) $$EX^2 < \infty \text{ implies } \lim_{s\uparrow 0} s^{\frac{1}{2}}Q(1-s) = 0 .$$

Since by (4.5) and (4.6)

(5.7) $$\lim_{\lambda\to\infty} \liminf_{n\to\infty} P\{\Omega_n(\lambda)\} = 1,$$

we obtain

(5.8) $$\Delta_{m,3}^{(1)} = o_P(1) .$$

Similarly, on $\Omega_n(\lambda)$,

$$\Delta_{m,2}^{(1)} \leq \int_0^{\lambda/n} |\hat{B}_m^*(s)|dQ(s) ,$$

and

$$E\int_0^{\lambda/n} |\hat{B}_m^*(s)|dQ(s) \leq |\int_{1/m}^{\lambda/m} s^{\frac{1}{2}}dQ(s)|$$

$$\leq C\, n^{-\frac{1}{2}} \sup_{0<s\leq(\lambda/n)\vee(C_2/n)} Q(s)$$

by condition (2.6). Since this bound goes to zero, (5.7) again implies that

(5.9) $$\Delta_{m,2}^{(1)} = o_P(1) .$$

Finally, choose any $0 < \nu < 1/4$. Then by Proposition 1

$$\Delta_{m,1}^{(1)} = O_P((n\wedge m)^{-\nu}) \int_{U_{1,n}}^{U_{n,n}} (s(1-s))^{\frac{1}{2}-\nu}\, dQ(s) .$$

Therefore, for any fixed $\varepsilon > 0$, on $\Omega_n(\lambda)$ we have

$$\Delta_{m,1}^{(1)} \le O_P((n \wedge m)^{-\nu}) \int_0^{1-\frac{1}{n\lambda}} (1-s)^{\frac{1}{2}-\nu}\, dQ(s)$$

$$= O_P((n \wedge m)^{-\nu}) \left\{ \int_{1-\varepsilon}^{1-\frac{1}{n\lambda}} (1-s)^{\frac{1}{2}-\nu}\, dQ(s) + \int_0^{1-\varepsilon} (1-s)^{\frac{1}{2}-\nu}\, dQ(s) \right\}$$

$$= O_P((n \wedge m)^{-\nu}) \left\{ \frac{1}{n\lambda}^{\frac{1}{2}-\nu} Q(1-\frac{1}{n\lambda}) + (\frac{1}{2}-\nu) \int_{1-\varepsilon}^{1-\frac{1}{n\lambda}} (1-s)^{-\frac{1}{2}-\nu} Q(s)\, ds \right\}$$

$$+ o_P(1)$$

upon integrating by parts. Therefore, using (5.6) we see that on $\Omega_n(\lambda)$

$$\Delta_{m,1}^{(1)} = O_P((n \wedge m)^{-\nu}) \left\{ O(n^{-\nu}) o(1) + \sup_{0<s\le\varepsilon} s^{\frac{1}{2}} Q(1-s) \int_0^{1-\frac{1}{n\lambda}} (1-t)^{-1-\nu} dt \right\}$$

$$+ o_P(1)$$

$$= o_P(1) + O_P((n \wedge m)^{-\nu}) n^{\nu} \sup_{0<s\le\varepsilon} s^{\frac{1}{2}} Q(1-s).$$

Since $\varepsilon > 0$ is arbitrarily small, (5.6) again gives that $\Delta_{m,1}^{(1)} = o_P(1)$ on the event $\Omega_n(\lambda)$. But (5.7) then implies that $\Delta_{m,1}^{(1)} = o_P(1)$. This relation, (5.4), (5.5), (5.8) and (5.9) imply the lemma.

<u>Proof of Theorem 1.</u> First we define the sequence of Gaussian processes

$$\hat{Z}_F^{(m)}(t) = \left\{ -\int_{F(t)}^{1} \hat{B}_m(s)\, dQ(s) + M_F(Q(F(t)))\hat{B}_m(F(t)) \right\} / (1-F(t)).$$

We get from Lemma 3, the first statement of Proposition 2 (applied with $q = 1$) and the Glivenko-Cantelli theorem that

$$\text{(5.10)} \qquad \sup_{0\le t\le T} \left| \hat{z}_{m,n}(t) - \hat{Z}_F^{(m)}(t) \right| = o_P(1), \quad F(T) < 1.$$

An elementary calculation yields

$$\sup_{0\le t<\infty} |(1-G_{m,n}(F(t)))\hat{z}_{m,n}(t) - (1-F(t))\hat{Z}_F^{(m)}(t)|$$

$$\le \sup_{0\le t<\infty} |\int_{F(t)}^{1} (\hat{B}_m(s) - \alpha_{m,n}(s))dQ(s)|$$

$$+ \sup_{0\le t<\infty} M_F(Q(F(t)))|\alpha_{m,n}(F(t)) - \hat{B}_m(F(t))|$$

$$+ \sup_{0\le t<\infty} |\alpha_{m,n}(F(t))||M_n(Q(F(t)))-M_F(Q(F(t)))|.$$

Here the first term converges to zero in probability by Lemma 3. The second term is not greater than

$$\sup_{0\le s<1} M_F(Q(s))|\alpha_{m,n}(s)-\hat{B}_m(s)|.$$

In the proof of (2.3) (Theorem 4.1 in Cs-Cs-H (1986)) it is shown that if $EX^2 < \infty$ then the function $M_F(Q(s)) = 1/q^o(s)$ is nondecreasing and square integrable on (0,1). This implies that $q^o(s)$ is a Chibisov-O'Reilly function, and hence Proposition 2 ensures that the last expression goes to zero in probability.

Finally, let $0 < \tau < 1$ be any fixed number. Then for large enough n the third term is almost surely less than or equal to

$$\sup_{0\le s\le\tau} |\alpha_{m,n}(s)||M_n(Q(s))-M_F(Q(s))|$$

$$+ \sup_{\tau\le s<U_{n,n}} |\alpha_{m,n}(s)|M_F(Q(s))|\frac{M_n(Q(s))}{M_F(Q(s))} - 1| .$$

Here the first term goes to zero in probability by (2.4) and Proposition 2 for any $0 < \tau < 1$. On the other hand, it is easy to show by (4.3) and (4.4) that

$$\sup_{\tau \leq s < U_{n,n}} \frac{M_n(Q(s))}{M_F(Q(s))} = \sup_{\tau \leq s < U_{n,n}} \frac{1-F(Q(s))}{1-G_n(F(Q(s)))} \frac{\int_{Q(s)}^{Q(U_{n,n})} (1-G_n(F(x)))dx}{\int_{Q(s)}^{\infty} (1-F(x))dx}$$

$$= O_P(1) .$$

We know that $q^o(s) = 1/M_F(Q(s))$ is square integrable on $(0,1)$, therefore for any $\varepsilon > 0$

$$\lim_{\tau \uparrow 1} \limsup_{n \to \infty} P\{ \sup_{\tau \leq s < U_{n,n}} M_F(Q(s)) |\alpha_{m,n}(s)| > \varepsilon\} = 0.$$

We proved that

$$(5.11) \qquad \sup_{0 \leq t < \infty} |\hat{z}_{m,n}(t) - \hat{z}_F^{(m)}(t)| = o_P(1).$$

Using now Lemma 4.4.4 of M. Csörgő and Révész (1981) (cf. also De Acosta (1982)), (5.1) and (5.11) imply Theorem 1.

<u>Proof of Theorem 2.</u> By a simple rearrangement we see that

$$\hat{t}_{m,n}(s) = -\int_0^s \alpha_{m,n}(t)dQ(t) + m^{\frac{1}{2}} \int_{U_n(s)}^{U_{m,n}(s)} (1-t)dQ(t) - \Delta_m^{(3)}(s) - \Delta_m^{(4)}(s),$$

where

$$\Delta_m^{(3)}(s) = \int_s^{U_{m,n}(s)} \alpha_{m,n}(t)dQ(t) \text{ and } \Delta_m^{(4)}(s) = \int_{U_n(s)}^{U_{m,n}(s)} \alpha_n(t)dQ(t).$$

Therefore, if

$$\hat{T}_F^{(m)}(u) = -\int_0^u \hat{B}_m(s)\,dQ(s) - \frac{1-u}{f(Q(u))}\hat{B}_m(u), \quad 0 \le u \le 1,$$

then

$$(5.12) \qquad \Delta_m = \sup_{0\le s\le 1} |\hat{t}_{m,n}(s) - \hat{T}_F^{(m)}(s)| \le \Delta_m^{(1)} + \ldots + \Delta_m^{(5)},$$

where

$$(5.13) \qquad \Delta_m^{(1)} = \sup_{0\le s\le 1} \left|\int_0^s \{\hat{B}_m(t) - \alpha_{m,n}(t)\}\,dQ(s)\right| = o_P(1)$$

by Lemma 3,

$$\Delta_m^{(2)} = \sup_{0\le s\le 1} \left|\frac{1-s}{f(Q(s))}\hat{B}_m(s) - m^{\frac{1}{2}}\int_{U_n(s)}^{U_{m,n}(s)} (1-s)\,dQ(s)\right|,$$

and

$$\Delta_m^{(i)} = \sup_{0\le s\le 1} |\Delta_m^{(i)}(s)|, \quad i = 3,4.$$

First we consider $\Delta_m^{(4)}$. By Lemma 2 we have

$$\Delta_m^{(4)} \le \sup_{0\le s\le 1} \left|\int_{U_n(s)}^{U_{m,n}(s)} B_n(t)\,dQ(t)\right| + o_P(1).$$

In Lemma 5.1 of Cs-Cs-H (1986) it is proved that if $EX^2 < \infty$ and Q is continuous on $[0,1)$, and if B denotes any Brownian bridge, then the stochastic process

$$\int_0^s B(t)\,dQ(t)$$

is almost surely continuous on $[0,1]$. Therefore,

$$\lim_{h\downarrow 0} \limsup_{n\to\infty} P\left\{ \sup_{|s-t|\leq h} \left|\int_s^t B_n(u)\,dQ(u)\right| > \varepsilon \right\}$$

$$= \lim_{h\downarrow 0} \Pr\left\{ \sup_{|s-t|\leq h} \left|\int_s^t B(u)\,dQ(u)\right| > \varepsilon \right\} = 0$$

for any $\varepsilon > 0$. Since by the Glivenko-Cantelli theorem

$$\sup_{0\leq s\leq 1} |U_{m,n}(s)-U_n(s)| = o_P(1),$$

this implies that

(5.14) $$\Delta_m^{(4)} = o_P(1).$$

Using Lemma 4 instead of Lemma 2, exactly the same argument gives

(5.15) $$\Delta_m^{(3)} = o_P(1).$$

Relations (5.12), (5.13), (5.14) and (5.15) show that we only need to prove that

(5.16) $$\Delta_m^{(2)} = o_P(1).$$

Pick ε such that $0 < \varepsilon < 1-\varepsilon < 1$. We have

(5.17)
$$\begin{aligned}\Delta_m^{(2)} &\leq \sup_{0\leq s\leq\varepsilon,\,1-\varepsilon\leq s\leq 1} \left|\frac{1-s}{f(Q(s))}\hat{B}_m(s)\right| \\ &+ \sup_{0\leq s\leq\varepsilon,\,1-\varepsilon\leq s\leq 1} \left|m^{\frac{1}{2}}\int_{U_n(s)}^{U_{m,n}(s)} (1-t)\,dQ(t)\right| \\ &+ \sup_{\varepsilon\leq s\leq 1-\varepsilon} \left|\frac{1-s}{f(Q(s))}\hat{B}_m(s) - m^{\frac{1}{2}}\int_{U_n(s)}^{U_{m,n}(s)} (1-t)\,dQ(t)\right| \\ &:= \Delta_{m,1}^{(2)} + \Delta_{m,2}^{(2)} + A_{m,n}(\varepsilon,1-\varepsilon).\end{aligned}$$

In the course of the proof of (2.13) (Theorem 6.2 in Cs-Cs-H (1986)) it was shown that

(5.18) $$\Delta^{(2)}_{m,1} = o_P(1).$$

A simple manipulation results in

$$\Delta^{(2)}_{m,2} \leq \sup_{0\leq s\leq \varepsilon,\, 1-\varepsilon\leq s\leq 1} |m^{\frac{1}{2}}\{H_F^{-1}(U_{m,n}(s))-H_F^{-1}(s)\}|$$

$$+ C_1^{-\frac{1}{2}} \sup_{0\leq s\leq \varepsilon,\, 1-\varepsilon\leq s\leq 1} |m^{\frac{1}{2}}\{H_F^{-1}(U_n(s))-H_F^{-1}(s)\}|$$

$$:= A^{(2)}_{m,n}(\varepsilon) + A^{(2)}_n(\varepsilon) ,$$

where, in the last step, we used condition (2.6). Again, in the proof of (2.13) it was shown, using the linearity properties in (4.6) and condition (2.12), that

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P\{A^{(2)}_n(\varepsilon) > \delta\} = 0$$

for any $\delta > 0$. We emphasize that this is the place where we require condition (2.12). Using now (4.17) instead of (4.6), exactly the same proof shows that under conditions (2.6) and (2.12) we have

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P\{A^{(2)}_{m,n}(\varepsilon) > \delta\} = 0$$

for any $\delta > 0$. Therefore,

(5.19) $$\Delta^{(2)}_{m,2} = o_P(1).$$

Finally we consider $A_{m,n}(\varepsilon,1-\varepsilon)$. Noting that

$$m^{\frac{1}{2}} \int_{U_n(s)}^{U_{m,n}(s)} (1-t)\,dQ(t) = m^{\frac{1}{2}}\{H_F^{-1}(U_{m,n}(s)) - H_F^{-1}(U_n(s))\}$$

and that $dH_F^{-1}(s)/ds = (1-s)/f(Q(s))$, a one-term Taylor expansion gives

$$\begin{aligned} A_{m,n}(\varepsilon,1-\varepsilon) &\le \sup_{\varepsilon \le s \le 1-\varepsilon} \left| \frac{1-\tau_{m,n}(s)}{f(Q(\tau_{m,n}(s)))} \beta_{m,n}(s) - \frac{1-s}{f(Q(s))} \hat{B}_m(s) \right| \\ &\le \sup_{\varepsilon \le s \le 1-\varepsilon} \frac{1-s}{f(Q(s))} \sup_{\varepsilon \le s \le 1-\varepsilon} |\beta_{m,n}(s) - \hat{B}_m(s)| \\ &\quad + \sup_{\varepsilon \le s \le 1-\varepsilon} \left| \frac{1-\tau_{m,n}(s)}{f(Q(\tau_{m,n}(s)))} - \frac{1-s}{f(Q(s))} \right| \sup_{\varepsilon \le s \le 1-\varepsilon} |\beta_{m,n}(s)|, \end{aligned} \tag{5.20}$$

where

$$U_{m,n}(s) \wedge U_n(s) \le \tau_{m,n}(s) \le U_{m,n}(s) \vee U_n(s).$$

The latter inequalities imply via the Glivenko-Cantelli theorem that

$$\sup_{0 \le s \le 1} |\tau_{m,n}(s) - s| = o_P(1).$$

Thus, since $f(Q(\cdot))$ is uniformly continuous on $[\varepsilon,1-\varepsilon]$ and since by Proposition 2, $\sup\{|\beta_{m,n}(s)| : \varepsilon \le s \le 1-\varepsilon\}$ has a limiting distribution, the second term on the right side of (5.20) converges to zero in probability. The first term goes to zero in probability by Proposition 2 for any fixed ε such that $0 < \varepsilon < 1-\varepsilon < 1$. This, together with (5.17), (5.18) and (5.19) implies that $\Delta_m = o_P(1)$, and therefore (2.15) follows from (5.2) and Lemma 4.4.4 of M. Csörgő and Révész (1981).

To prove the second statement in (2.16), we note that

$$\hat{s}_{m,n}(y) = \mu^{-1}\,\hat{t}_{m,n}(y) - \mu^{-2}H_F^{-1}(y)\,\hat{t}_{m,n}(1)$$

$$+\,\hat{t}_{m,n}(y)\{(1/\hat{H}_{m,n}^{-1}(1))-(1/H_F^{-1}(1))\}$$

$$+\,\mu^{-1}H_F^{-1}(y)t_n(1)\{(1/H_F^{-1}(1))-(1/\hat{H}_{m,n}^{-1}(1))\}$$

for any $0 \le y \le 1$. Hence the second statement is immediate.

Proof of Theorem 3. The proofs of Theorems 1 and 2 followed the general outlines of the proofs of (2.2), (2.3) and (2.13) given in Cs-Cs-H (1986). In a similar manner, the proof of Theorem 3 is obtained by performing steps very much like those carried out by Cs-Cs-H (1986) (cf. the proofs of Theorems 10.2 and 11.2) to establish (2.18), but now basing our proof on the techniques just developed for the bootstrapped uniform empirical and quantile processes.

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Table 1. Average deviation in Critical Value					
size level	10	20	50	100	200
.20	.189	.191	.167	.175	.178
	(.018)	(.037)	(.019)	(.009)	(.007)
.10	.206	.213	.190	.196	.197
	(.024)	(.044)	(.024)	(.012)	(.011)
.05	.215	.230	.209	.215	.214
	(.027)	(.046)	(.031)	(.015)	(.015)

FIGURE CAPTIONS

Figure 1. Tractor brakes with constant width bands: (a) total time on test transform, (b) scaled total time on test, and (b) Lorenz curve. Dash = curve estimate, solid = 90% bootstrapped confidence bands, dot = curve for exponential.

Figure 2. Tractor brakes with variable width bands based on estimated standard error. (a) total time on test transform, (b) scaled total time on test, and (b) Lorenz curve. Dash = curve estimate, solid = 90% bootstrapped confidence bands on [0.1,0.9], dot = curve for exponential.

Figure 1. Tractor Brakes with Constant Bands

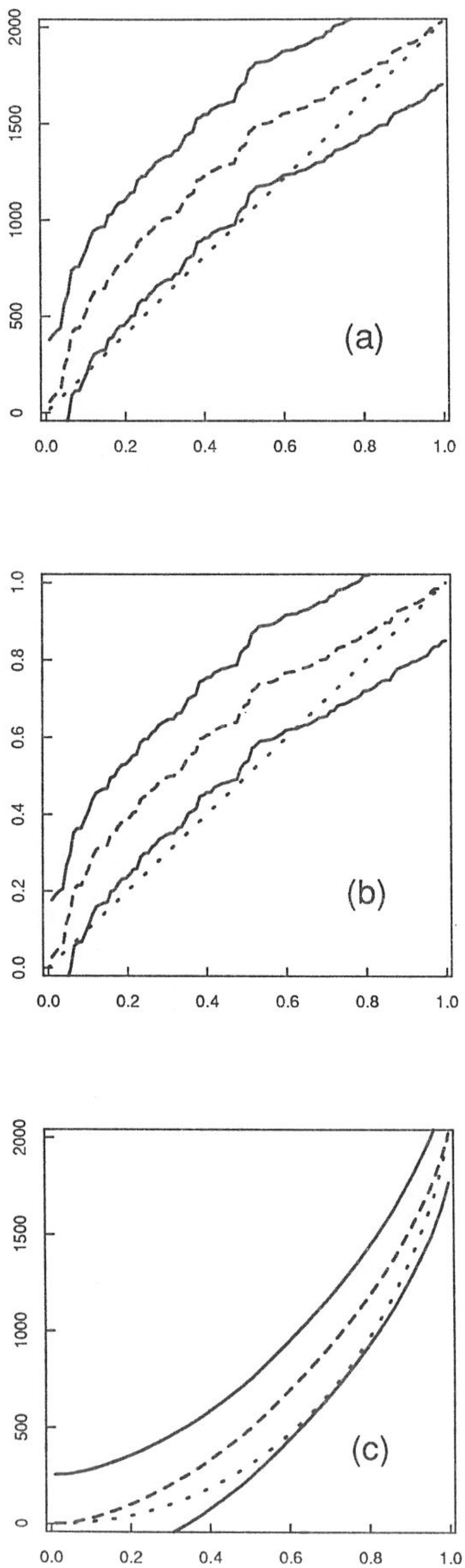

Figure 2. Tractor Brakes with Variable Bands