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SMALL SAMPLE PROPERTIES OF SPATIAL AUTOCORRELATIONS

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SUMMARY

Spatial patterns arise in various contexts such as agricultural field experiments, geological explorations and epidemic studies. We focus on data collected in regular grids, or contiguous quadrats, and examine the small sample properties of two tests for autocorrelation. We compare the powers of likelihood ratio statistics based on simultaneous autoregressive and simultaneous moving average models when the wrong model is chosen to the power of Moran's I statistic. An example from plant pathology is presented.

Keywords: LIKELIHOOD RATIO; MORAN'S I TEST; NEWTON-RAPHSON; SIMULTANEOUS AUTOREGRESSIVE; SIMULTANEOUS MOVING AVERAGE.

1. Introduction

Spatial patterns arise in various contexts such as agricultural field experiments, geological explorations and epidemic studies. In this paper, we focus on data collected in regular grids, or contiguous quadrats, and examine the small sample properties of two tests for autocorrelation.

We consider simple models which assume that measurements in contiguous quadrats may be correlated in some specific manner. Well known models include simultaneous autoregression (Whittle, 1954), conditional autoregression (Bartlett, 1971; Besag, 1974), simultaneous moving average (Haining, 1978), and error-in-variables (Besag, 1977). See Cliff and Ord (1981) and Ripley (1981) for further references.

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Another class of models assume that there is no spatial correlation, and that any spatial pattern can be explained by a smooth signal plus independent noise. The signal can ideally be modeled by further knowledge of the spatial substrate, such as water and soil factors in an agricultural experiment. However, lacking such information, one models the signal as a smooth surface (Green, Jennison and Seheult, 1985; Green, 1985; Green and Yandell, 1985; O'Sullivan, Yandell and Raynor, 1984). This class of models is not identifiable from spatial correlation models unless one replicates the experiment. Such smooth, or thin plate spline, models are not discussed further here.

The presence of spatial autocorrelation can be tested by using Moran's I statistic (Moran, 1950). Cliff and Ord (1975, 1981) have shown that under different types of rectangular grids Moran's I statistic was more powerful among several similar statistics. Their simulations were done on the first order simultaneous autoregressive model (SAR). On the other hand, Haining (1978) has shown that the likelihood ratio test (LR) was more powerful than I in the first order simultaneous moving average (SMA) model when the underlying spatial structure was generated from SMA.

The purpose of this paper is to determine the small sample power of Moran's I statistic in comparison with the likelihood ratio (LR) statistic when the underlying model is misspecified. We consider only the first order simultaneous autoregression (SAR) and first order simultaneous moving average (SMA) models for a rectangular grid design. That is, if the underlying model is SAR we use the LR for SMA, and vice versa. It is known that LR test is better than the I test when sample size is large and the underlying model is correct. Since the I test does not assume any underlying model, we want to find out how important this is for small samples when the model is misspecified.

2. Notation and Models

Consider a rectangular $m \times n$ grid, with $\mathbf{Y} = \{Y_i\}, i=1, \cdots, mn$ being the random vector of realised values for a spatial process. Without loss of generality let $E(\mathbf{Y})=0$. We define a simultaneous autoregressive model (SAR) as

$$\mathbf{Y} = \mathbf{G}\,\mathbf{Y} + \mathbf{E}\,, \quad \text{or} \quad (\mathbf{I}-\mathbf{G})\mathbf{Y} = \mathbf{E}\,,$$

where $\mathbf{G}$ is a $mn \times mn$ matrix containing the autocorrelation structure and $\mathbf{E}$ is distributed as $MVN(0, \sigma^2\mathbf{I})$. Note that

$$Var(\mathbf{Y}) = \sigma^2 \left[(\mathbf{I}-\mathbf{G}')(\mathbf{I}-\mathbf{G})\right]^{-1} .$$

A simultaneous moving average (SMA) model resembles SAR, with

$$\mathbf{Y} = (\mathbf{I}+\mathbf{H})\,\mathbf{E} ,$$

where **E** is distributed as $MVN(0,\sigma^2\mathbf{I})$. Thus

$$Var(\mathbf{Y}) = \sigma^2 (\mathbf{I}+\mathbf{H})(\mathbf{I}+\mathbf{H}') .$$

The matrices **G** and **H** can be parameterized to first order as

$$\mathbf{G} = \rho\mathbf{W} \text{ and } \mathbf{H} = \theta\mathbf{W}$$

with $\mathbf{W} = \{W_{ij}\}$, the $mn \times mn$ matrix of indicators for nearest neighbours.

2.1 Moran's I statistic

Moran's I statistic is defined by

$$I = \frac{mn \ \mathbf{Y}'\mathbf{W}\mathbf{Y}}{(\mathbf{1}'\mathbf{W}\mathbf{1})\,(\mathbf{Y}'\mathbf{Y})}$$

where **1** is a vector of 1's , **Y** is adjusted by the sample mean, and **W** is the matrix of nearest neighbour weights mentioned above. This statistic is widely used in testing the existence of spatial autocorrelation. Under either the SAR model with ρ=0 or SMA with θ=0, this statistic has mean $E(I) = -1/(mn-1)$ and variance

$$Var(I) = \frac{(mn)^2 \sum\sum (W_{ij}+W_{ji})^2 - mn \sum (W_{i.}+W_{.i})^2 + (\sum\sum W_{ij})^2}{((mn)^2-1)\,(\sum\sum W_{ij})^2} - (mn-1)^{-2}$$

See Cliff and Ord (1981) and Sen (1976) for details and results on asymptotic normality. Ali (1984) presents approximate methods for the computation of power when the sample size is small for the related Durbin-Watson statistic; these methods can be adapted to use with Moran's I statistic.

We focus in the remainder of this paper on the "rook's case", in which the nearest neighbors to a quadrat are the 4 quadrats with common edges, and the weights W_{ij} are 0 or 1 indicators. There is a problem as to how to assign weights at the perimeter of the grid. One solution is the torus lattice (Besag, 1974) where each outer edge is linked to the opposite side. Another solution is to reduce attention in the original grid to interior quadrats only. The third solution is to treat the weights unevenly, allowing for the exact number of neighbors for perimeter quadrats.

Each proposed method has its drawbacks. The torus lattice assumption is hardly satisfied in nature. The interior points approach loses degrees of freedom, particularly for small grids. The exact method can create numerical problems in estimation when row sums of **W** are not equal.

2.2 Likelihood Ratio test

The LR test requires the maximum likelihood (ML) estimate of the specified model under null and alternative. Suppose there are two sets of hypothesis of interest.

$$H_o\colon\ \mathbf{Y} = \mathbf{E}$$

$$H_1\colon\ \mathbf{Y} = (\mathbf{I}-\rho\mathbf{W})^{-1}\mathbf{E}$$

$$H_2\colon\ \mathbf{Y} = (\mathbf{I}+\theta\mathbf{W})\mathbf{E}$$

where **E** is distributed as $MVN(0,\sigma^2\mathbf{I})$. The log likelihood is

$$\log L(Y) = \left(\frac{-mn}{2}\right)\log(2\pi\sigma^2) - \tfrac{1}{2}\log|\mathbf{V}| - \frac{\mathbf{Y}'\mathbf{V}^{-1}\mathbf{Y}}{2\sigma^2}$$

with **V** under the three models being

$$H_0\colon\ \mathbf{V} = \mathbf{I}$$

$$H_1\colon\ \mathbf{V} = (\mathbf{I}-\rho\mathbf{W}')(\mathbf{I}-\rho\mathbf{W})^{-1}$$

$$H_2\colon\ \mathbf{V} = (\mathbf{I}+\theta\mathbf{W})(\mathbf{I}+\theta\mathbf{W}')$$

In order to have stationarity in the SAR model or invertibility of the SMA model, $|\rho|<1/4$ or $|\theta|<1/4$, respectively. The likelihood ratios that we consider are:

$$\lambda_1 = \frac{1}{|\mathbf{I}-\hat{\rho}WW|}\left\{\frac{Y'(I-\hat{\rho}W')(I-\hat{\rho}W)Y}{\mathbf{Y}'\mathbf{Y}}\right\}^{mn/2}$$

where $\hat{\rho}$ is the maximum likelihood estimate of ρ under H_1,

$$\lambda_2 = |\mathbf{I}+\hat{\theta}WW|\left\{\frac{Y'[(I+\hat{\theta}W)(I+\hat{\theta}W')]^{-1}Y}{\mathbf{Y}'\mathbf{Y}}\right\}^{mn/2}$$

where $\hat{\theta}$ is the maximum likelihood estimate of θ under H_2. The determinant and the inverse can be handled easily by expressing $\mathbf{W} = \mathbf{P}'\mathbf{D}\mathbf{P}$, where **D** is the diagonal matrix containing eigenvalues of **W** and **P** is its orthonormal decomposition. Numerical estimation of ρ and θ is done by Newton Raphson methods where the log likelihood is expressed in terms of ρ and θ. That is the ML estimate of σ^2 in terms of $\hat{\rho}$ or $\hat{\theta}$ is used in the likelihood function.

3. Simulation Procedure and Results

Grids of edge size 5 to 10 were simulated with $\sigma=1$ and correlation parameter, θ for the SAR model or ρ for the SMA model, having values between -0.25 and 0.25 in steps of 0.05. For each grid size and parameter combination, 1000 Monte Carlo simulations were performed. Each simulation yielded an I test statistic and an LR test statistic for the "wrong" model. That is, the LR statistic for the SMA model was computed for data drawn from an SAR distribution, and vice versa. Empirical power was calculated for $\alpha=0.05$ and 0.01. The extreme tails of the power curves were not computed due to numerical problems. Results from size of 1% case resemble the 5% case and will not be presented here.

Figures 1 to 4 summarise the simulation outcomes. Figures 1 and 2 show how the powers of the I and LR tests change with increasing grid size, while Figures 3 and 4 contrast the difference of power between the I test and the *LR* test for the smallest and the largest grid. Odd numbered figures represent the results from the SAR model fitted to the SMA generated data. The even numbered figures represent the other 'wrong' model.

Note that in both models, the power increases as grid size increases. For instance, in the SAR fit to the SMA model, the power of the I test for a 5×5 grid at $\theta=-0.1$ is 0.2701, while the power for a 10×10 grid at the same θ is 0.8054 (see Table 1 and Figure 1). The increase in power is small for moderately large grids, say 8×8. In another words, the improvement in power of either test is small in comparison to the extra cost of collecting more data when the grid is about 7×7 or 8×8. Note also that the size of the tests become more accurate as grid size increases. The LR test appears to be more powerful and seems preferred when the grids are small. When grids are large, either test can be used. See Table 2 for tests of data simulated from the SAR model.

Another common feature in these two misclassified models is that the LR power curves are shifted to the right, with the minimum occuring at a point slightly larger than zero. This shift, noted by Brandsma and Ketellapper (1979), appears to be due to bias in estimating the correlation parameter. which is borne out by the fact that the shift disappears if one plots the power against the estimated parameter rather than against the true parameter. The power curves for the I statistic show no such bias.

The fact that the SMA model appeared more powerful than the SAR is explained (numerically) by examining the likelihood ratio statistics under these two hypothesised models. In all the simulations, the likelihood ratio statistics obtained under SAR were larger than the ones

obtained under SMA on the average. Hence, the approximate χ^2 statistic, $-2\log(LR)$, had the reverse inequality; thus the SMA likelihood ratio statistic was more extreme. This was true even at H_0. As a result, the SMA result has a size larger than 5 percent.

Additional simulations on rectangular grids such as 3×12 and 2×32 indicated that power varies somewhat with shape, being slightly lower when edge elements are many. From these simulations, there is no evidence that the 'wrong' model is less powerful than the correct model.

4. An Example

A study of inoculum levels of Verticillium dahliae by Philippe Nicot, part of his doctoral dissertation in the Plant Pathology Department, University of Wisconsin–Madison, provided us with a number of square grids of various sizes. There were 44 study areas(grids) and sizes ranged from 5×5 to 20×20. Each quadrat of a grid is an observed inoculum levels which is an aggregated variable. Hence a square root transformation is needed before the calculations. Our interest is to compare Moran's I test and the likelihood test. In several nonempty grids selected, both tests yielded the same result. For example,

grid size	I	p-value	LR(SAR)	p-value	θ
10x10	0.3680	<0.001	0.000023	<0.005	0.1425
5x5	-0.0106	¯0.3594	0.9977	¯0.95	-0.0055

This was the most marked difference between the I and LR tests noted. In most cases autocorrelation was clearly evident or absent. Little difference was seen between tests based on the SAR or the SMA.

5. Conclusion

(1) As grid size increases, power increases, but the increase tends to be small when the grid is already large. For an instance in the likelihood test, the power increase from grid 6×6 to 7×7 is around 0.15 and the gain from grid 9×9 to 10×10 is around 0.05, when the underlying parameter equals to 0.1.

(2) When sample size is small, the LR approach appears to yield a more powerful test even if the model is wrong. Nevertheless, the I test performs just as well when the grid size is large, say

larger than 7×7.

(3) When the LR test is used, the simultaneous autoregressive model is preferred. From numerical study, the maximum likelihood estimates under SAR showed less bias than the ones under SMA.

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TABLE 1. Simulated SMA data.

Table 1(a). Moran's *I* test for SMA data.

θ	5	6	7	8	9	10
-0.2000	0.6698	0.8616	0.9053	0.9394	0.8922	0.7549
-0.1000	0.2701	0.3644	0.5000	0.5942	0.7410	0.8054
0.	0.0590	0.0480	0.0450	0.0428	0.0579	0.0491
0.1000	0.2853	0.4180	0.4880	0.6144	0.7337	0.8022
0.2000	0.7182	0.8536	0.9371	0.9454	0.9109	0.8248

Table 1(b). LR(SAR) test for SMA data.

θ	5	6	7	8	9	10
-0.2000	0.8025	0.9329	0.9403	0.9507	0.8942	0.7569
-0.1000	0.3514	0.4655	0.5900	0.6667	0.7957	0.8538
0.	0.0690	0.0450	0.0460	0.0459	0.0621	0.0428
0.1000	0.2002	0.3160	0.4180	0.5569	0.6916	0.7768
0.2000	0.6435	0.8243	0.9361	0.9454	0.9109	0.8248

TABLE 2. Simulated SAR data.

Table 2(a). Moran's *I* test for SAR data.

ρ	5	6	7	8	9	10
-0.2000	0.7663	0.9302	0.9726	0.9930	0.9960	0.9860
-0.1000	0.2465	0.3440	0.4610	0.5810	0.6750	0.7760
0.	0.0624	0.0511	0.0420	0.0530	0.0540	0.0530
0.1000	0.2091	0.3233	0.4050	0.5580	0.6740	0.7500
0.2000	0.6878	0.8490	0.9628	0.9880	0.9980	0.9980

Table 2(b). LR(SMA) test for SAR data.

ρ	5	6	7	8	9	10
-0.2000	0.8457	0.9559	0.9837	0.9899	0.9950	0.9790
-0.1000	0.3479	0.4594	0.5410	0.6510	0.7350	0.8270
0.	0.0986	0.0641	0.0610	0.0690	0.0610	0.0540
0.1000	0.1515	0.2623	0.3520	0.5000	0.6440	0.7250
0.2000	0.5694	0.8075	0.9205	0.9600	0.9630	0.9720

FIGURE CAPTIONS

Figure 1. Power for simulated SMA data at α=0.05, grids with edge size 5 to 10, and θ from –0.2 to 0.2 in steps of 0.05. (a) Moran's *I* test; (b) LR test based on SAR model.

Figure 2. Power for simulated SAR data at α=0.05, grids with edge size 5 to 10, and ρ from –0.2 to 0.2 in steps of 0.05. (a) Moran's *I* test; (b) LR test based on SMA model.

Figure 3. Comparison of power for simulated SMA data at α=0.05, and θ from –0.2 to 0.2 in steps of 0.05. Moran's *I* test (solid) vs. LR test based on SAR model (dashed). (a) 5×5 grid; (b) 10×10 grid.

Figure 4. Comparison of power for simulated SAR data at α=0.05, and ρ from –0.2 to 0.2 in steps of 0.05. Moran's *I* test (solid) vs. LR test based on SMA model (dashed). (a) 5×5 grid; (b) 10×10 grid.

Figure 1 SAR fitted to SMA.

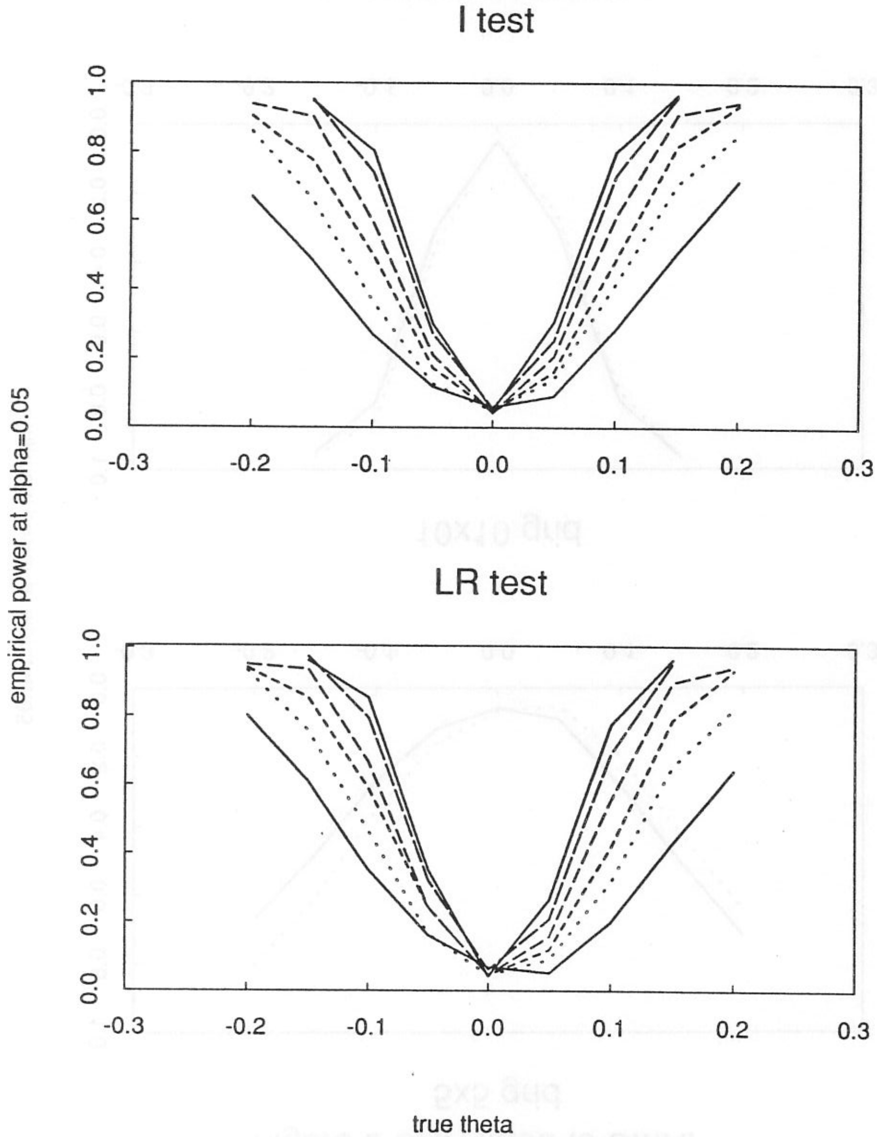

Figure 2 SMA fitted to SAR.

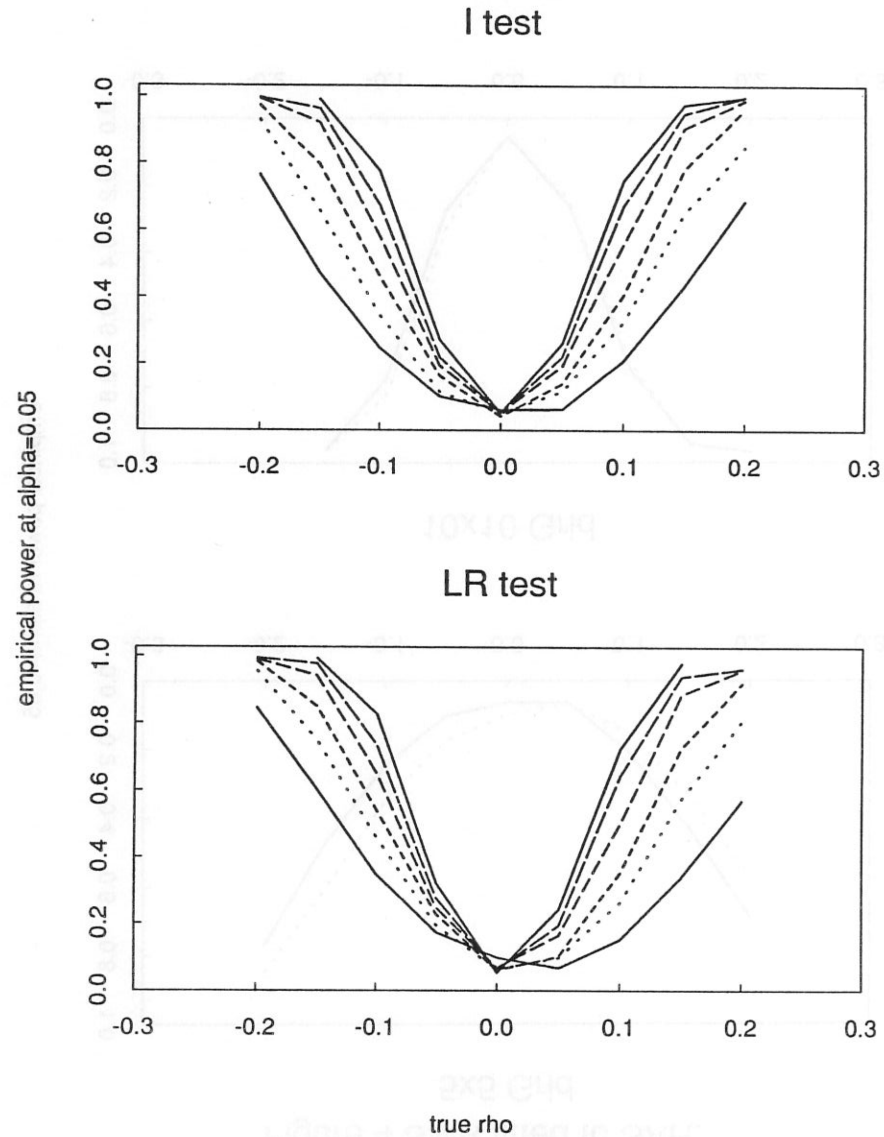

Figure 3 SAR fitted to SMA.

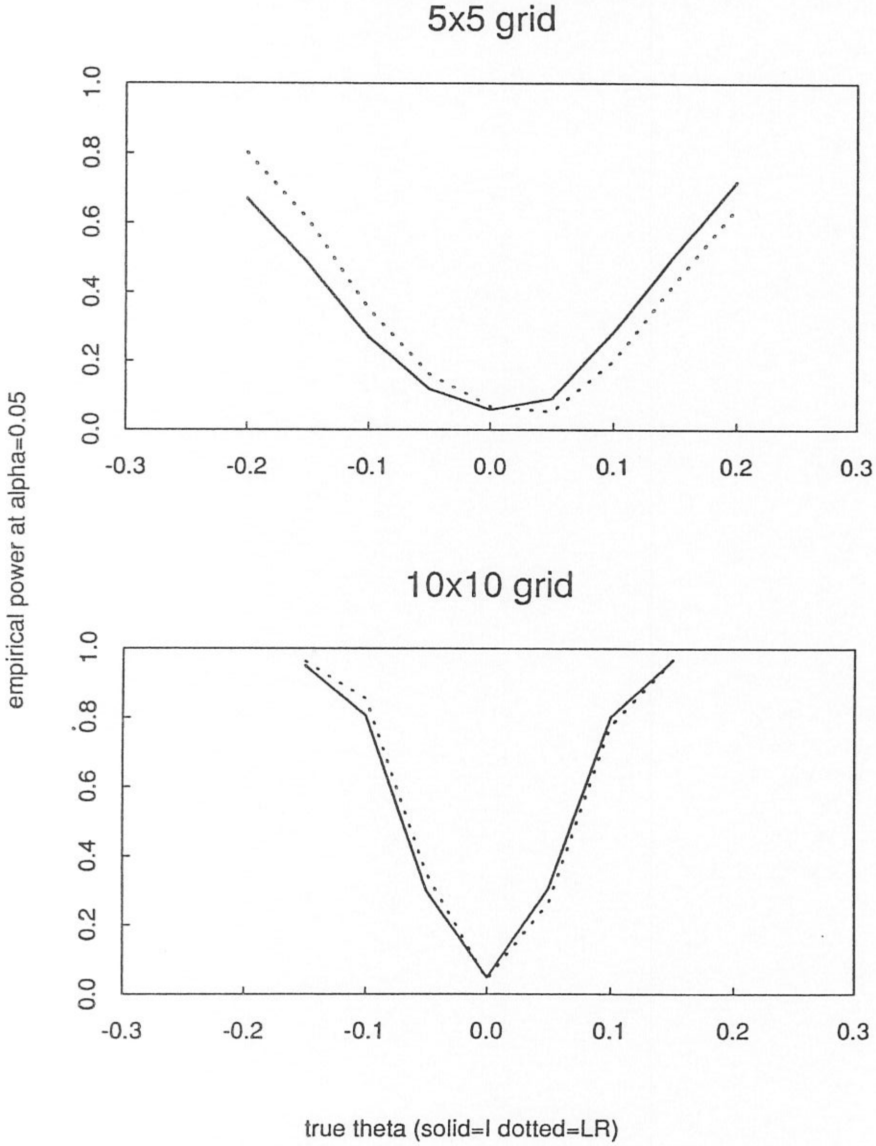

Figure 4 SMA fitted to SAR.