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AN APPROXIMATION OF THE K-FUNCTION FOR STRAUSS DISC PROCESSES

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Abstract

We extended some results of Isham (1984) that worked on an approximation for the K-function for multitype point processes, based on the assumptions that edge effects are negligible and interaction parameter is fairly weak, for the Strauss disc process. A simulation study showed that the approximation for the unmarked pairwise-interaction-type point processes seems reasonable, but does not work well for Strauss disc process. Moreover, this study led us to question the use of the K-function differences as a basis for comparison of Strauss disc type patterns.

Keywords: Reduced second moment measure; pairwise-interaction-type point process; marked point process

AMS 1991 Subject Classification: Primary 62M30
Secondary 60K40

1. Introduction

Spatial statistics methodology has been largely explored and used in recognition of spatial patterns characterizing a specific region. Many of the applications are in forestry and ecology but more recently other fields also started exploring spatial statistics techniques. Pairwise interaction point processes are frequently used to model and explain certain mechanisms involved in some practical problems. They are a special case of Gibbs point processes where the interaction function depends only on the distance between two points (Diggle et al. 1983, Cressie 1991, Ripley

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1988, Stoyan et al. 1987).

Various statistics have been created based on measures on neighborhoods, for example, the nearest neighbor distance. Authors like Ripley (1988), Stoyan et al. (1987), Upton and Fingleton (1985), Diggle et al. (1983) and Cressie (1991) describe some of these summary statistics and their properties for the case of a completely spatially random point process (CSR). Some of these summaries are one-dimensional and hence lead to a considerable loss of information. On the other hand, there are summary functions that give a picture about the departure from the CSR in the presence of inhibition or clustered patterns in a field. Two examples of such functions are K-function or reduced second moment measure and nearest-neighbor distance distribution function.

For marked point processes, there is some work also done in terms of summaries for goodness-of-fit or exploratory analysis, like the K-function version for marked point processes based on some mark correlation measure (Stoyan 1984). Pentinnen et al. (1992) used this function to describe spatial dependence of stem diameters, and crown lengths in a mixed birch-pine forest area. Goulard et al. (1996) present a general definition and derivation of the pseudo-likelihood function for marked Gibbs point processes. They derived the maximum pseudo-likelihood inference for two particular cases, bivariate Gibbs point process and Strauss disc process. They presented a simulation study for the bivariate Gibbs point process, and applied the method to a forestry example (same used in Penttinen et al. (1992)) of the Strauss disc pattern.

The Strauss disc process is a special case of marked pairwise interaction point processes (Badelley and Møller 1989, Geyer and Møller 1994, Goulard et al. 1996). However, little is known about properties of summaries with respect to this process, in particular about the reduced second moment measure. Any progress towards this matter would be of great help for infering and characterizing spatial patterns.

Isham (1984) derived approximations for some of the properties of a Strauss-type point process with two types of points and indicated how to generalize for N types of points, on the assumption that interactions between the points are fairly weak and that the boundary effects are negligible. In particular, expansions are obtained for the reduced second moment measure. In the present paper, approximations are derived for the K-function of a Strauss disc process on the same assumptions and using the same method of approximation as in Isham (1984).

1.1. Preliminaries in spatial point processes

In this section we present a brief review of the theory involved in spatial point processes. We go through some definitions based on some reference books such as Ripley (1988), Diggle (1983) and Cressie (1991).

In the following, let $\mathcal{X}$ be the Borel σ-algebra of the sample space $X \subset \Re^d$ for $d \geq 2$ and let ν be the Lebesgue measure on X. Recall that a measure μ is locally finite if $\mu(B) < \infty$ for all bounded sets $B \in \mathcal{X}$. A counting measure $\psi(B)$ is the number of events in B, that is, the number of subsets of X.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and let C be a collection of locally finite counting measures on $X \subset \Re^d$. On C define $\mathcal{N}$, the smallest σ-algebra generated by sets of the form $\{\psi \in C : \psi(B) = n\}$, for all $B \in \mathcal{X}$ and all $n \in \{0, 1, 2, ...\}$. Then a *spatial point process* Ψ on X is a measurable mapping of $(\Omega, \mathcal{A})$ into $(C, \mathcal{N})$. This random counting measure ψ on X is analogous to a random variable on $\Re$, and probabilities are computed from

$$P(\Psi \in Y) = P(w : \Psi(w) \in Y), \quad Y \in \mathcal{N}.$$

A spatial point process defined over $(\Omega, \mathcal{A}, P)$ induces a probability measure

$$\pi_\Psi(Y) \equiv P(\Psi \in Y) \; on \; (C, \mathcal{N}), \quad \text{for all} \quad Y \in \mathcal{N}.$$

A *spatial point pattern* ψ is a realization of a spatial point process Ψ.

For a point process Ψ and Borel set B, the number of points $\Psi(B)$ in B is a random variable with *first moment measure*

$$\kappa_\Psi(B) \equiv E(\Psi(B)) = \int_C \psi(B) \; \pi_\Psi(d\psi),$$

a measure on $(X, \mathcal{X})$. The *intensity* of a point process is defined by

$$\lambda(s) \equiv \lim_{\nu(ds) \to 0} \frac{\kappa_\Psi(ds)}{\nu(ds)},$$

provided the limit exists. Also, we can define the *second moment measure* by

$$\kappa_\Psi^{(2)}(B_1 \times B_2) \equiv E(\Psi(B_1)\Psi(B_2)) = \int_C \psi(B_1)\psi(B_2)\, \pi_\Psi(d\psi),$$

a measure on $(X^2, \mathcal{X}^{(2)})$, with $\mathcal{X}^{(2)}$ being the smallest σ-algebra formed by the product sets $B_1 \times B_2$ and $B_1, B_2 \in \mathcal{X}$. Then the *second-order intensity* is defined by

$$\lambda_2(s,u) \equiv \lim_{\substack{\nu(ds) \to 0 \\ \nu(du) \to 0}} \frac{\kappa_\Psi^{(2)}(ds \times du)}{\nu(ds)\, \nu(du)},$$

provided the limit exists.

1.2. The K-function

The *reduced second moment measure*, or K-function, is defined as

(1) $$K(t) = \frac{1}{\lambda} E \begin{pmatrix} \text{number of extra events within} \\ \text{distance } t \text{ of an arbitrary event} \end{pmatrix},\ t \geq 0,$$

where λ is the intensity of the point process. Under the assumption of CSR in $\Re^2$, $K(t) = \pi t^2$, while under regularity $K(t)$ tends to be less than πt^2 and under clustering $K(t)$ tends to be greater than πt^2. This paper concerns processes that are more regular than CSR.

The relation between the K-function and the second-order intensity is given by

(2) $$\lambda K(t) = \frac{d\pi^{d/2}}{\lambda \Gamma\left(1 + \frac{d}{2}\right)} \int_0^t u^{d-1}\, \lambda_2(u)\, du,\ t \geq 0,$$

where d is the dimension of the considered space and λ_2 is the second-order intensity of the point process defined.

Let $(x_1, ..., x_n)$ denote the $n \equiv n(A)$ locations of all events in a bounded convex study region A. Ripley (1979) uses an edge-corrected estimator for $K(t)$, defining

(3) $$\hat{K}(t) = \frac{1}{n\hat{\lambda}} \sum_{i=1}^{n} \sum_{j=1}^{n} I(\| x_i - x_j \| \leq t) \times k(x_i, x_j),$$

with $1/k(x,y)$ being the proportion of the circumference of the circle centered at x passing through y which is within A. For all the estimators involving $\hat{\lambda}$ we can use

$\hat{\lambda} = \frac{n}{|A|}$. According to Ripley (1981), the use of $\hat{\lambda}$ seems not to upset the unbiasedness of the estimator too much. Ripley (1979) also suggests that for some t_0 small enough, the bias in $\hat{K}(t)$ is negligible for $t \le t_0$.

1.3. Strauss disc processes

Random disc processes (Baddeley and Møller 1989) are special cases of the germ-grain model (Stoyan et al. 1987) which can be used to describe a pattern of randomly distributed discs (or spheres in higher dimensions), where discs (spheres) may overlap.

Consider a marked point process $\Psi_m = \{[x_i, r_i]\}$ with a finite number of points on a bounded space Ω and marks on $\Delta \subset [0, \infty)$, where the density is the Radon-Nikodym derivative with respect to the distribution of a homogeneous Poisson process (which, with no loss of generality, can be taken to have unit rate). Let $x = (x_1, ..., x_n)$, $r = (r_1, ..., r_n)$, and consider $[x_i, r_i]$ to be a disc with radius r_i centered at x_i. A general pairwise interaction process has density

$$f(x, r) = z\beta^n \prod_{i<j} \phi(\| x_i - x_j \|, r_i, r_j),$$

where $\phi : [0, \infty)^3 \to [0, \infty)$.

Our particular interest concerns the Strauss-like interaction function given by

$$\phi(u, r_i, r_j) = \begin{cases} \gamma & if \quad u < r_i + r_j \\ 1 & \text{else,} \end{cases}$$

for $0 \le \gamma \le 1$. This process is called a *Strauss disc process* and its density becomes

$$f(x, r) = z\beta^{n(x,r)}\gamma^{s(x,r)}, \tag{4}$$

where β is a parameter related to the intensity of the point process, γ is the interaction parameter, $n(x, r)$ is the number of points in Ψ_m and

$$s(x, r) = \sum_{\substack{i,j=1 \\ i<j}}^{n} \chi(\| x_i - x_j \|, r_i, r_j),$$

is the number of pairs of points x_i and x_j that are closer than $r_i + r_j$ apart in Ψ_m, with $\chi(u, r_i, r_j)$ being equal to 1, if $u \le r_i + r_j$, or 0 otherwise.

2. Approximation of the K-function

Approximations are derived for the K-function of a Strauss disc process, on the assumption that the interaction between points is fairly weak. That is, we consider the position of the centers of the discs as distributed close to the Poisson process, and that the boundary effects are negligible.

2.1. Isham's results for multitype point processes

Isham (1984) developed approximations for some of the properties of a Strauss-type spatial point process with two types of points (easily extended to N types of points, or a multitype process), on the assumption that the interactions between the points are fairly weak and that boundary effects are negligible. The last assumption will be true when the considered region is large enough and the discs are far away from the boundaries. Consider a marked point process $\{[x_i, r_i]\}$. We say $r_i = \delta_m$, with $\delta_m > 0$, if point x_i is of type m, $m = 1, \cdots, N$, and denote $\delta_{ml} = \delta_m + \delta_l$. The density function of the Strauss-multitype process is given by

$$f(x,r) = z \prod_{m=1}^{N} \beta_m^{n_m(x,r)} \prod_{1 \le m \le l \le N} \gamma_{ml}^{s_{ml}(x,r)}, \tag{5}$$

where $n_m(x,r)$ is the number of points of type m with $\sum_{m=1}^{N} n_m(x,r) = n$, and $s_{ml}(x,r)$ is the number of pairs of points of types m and l that are closer than δ_{ml}. Here $z, \beta_1, ..., \beta_N$, are positive constants. The author wrote

$$s_{ml}(x,r) = \sum_{i=1}^{n_m(x,r)} \sum_{j=1}^{n_l(x,r)} \chi^{(N)}(\| x_{mi} - x_{lj} \|, \delta_{ml}),$$

with x_{mi} being the ith point of type m and $\chi^{(N)}(u, \delta_{ml})$ equal to 1, if $u \le \delta_{ml}$ or 0, otherwise.

For the existence of the process it is necessary that the joint densities of the points specified by (5) can be appropriately normalized up to a constant z. This implies that within each type the interactions are inhibitory, that is, $0 \le \gamma_{mm} \le 1$. Between types some attraction may be possible but usually only to a limited extent. Based on the assumptions stated previously, Isham considers $\gamma_{mm} = 1 - g_m \ \epsilon$ and $\gamma_{ml} = 1 + d_{ml} \ \epsilon$, for $m, l = 1, \cdots, N$, $m \neq l$, for some small $\epsilon > 0$, where $g_m \geq 0$ and d_{ml} are arbitrary subject to the existence of the process.

The type m intensity is then approximated to first order by

$$\lambda_m = \beta_m \ \{1 - \epsilon \ (g_m \ \beta_m \ \pi \ \delta_{mm}^2 - \sum_{l \neq m} d_{ml} \ \beta_l \ \pi \ \delta_{ml}^2) + O(\epsilon^2)\}$$

and the joint intensity of a type m point at ξ and a type l point at η is approximated by

$$\lambda_{ml}(\xi, \eta) = \lambda_m \lambda_l \ \{1 + \epsilon \ d_{ml} \ \chi^{(N)}(\xi, \eta, \delta_{ml}) + O(\epsilon^2)\}.$$

The joint intensities λ_{ml} can be integrated to determine second-order properties of the multitype point process. In particular, the reduced second moment measure, following (1) and (2), is defined for $t \geq 0$ by

$$\text{(6)} \qquad \begin{aligned} K_{ml}(t) &= \frac{1}{\lambda} E \left(\begin{array}{c} \text{number of type } l \text{ points within a} \\ \text{distance } t \text{ of a given type } m \text{ point} \end{array} \right) \\ &= \int_0^t 2\pi s \ \frac{\lambda_{ml}(s)}{\lambda_m \lambda_l} \ ds \end{aligned}$$

where $\lambda_{ml}(\xi, \eta) = \lambda_{ml}(\| \ \xi - \eta \ \|)$. A first order approximation is

$$\text{(7)} \qquad K_{ml}(t) \approx \begin{cases} \gamma_{ml} \ \pi \ t^2 & if \quad t \leq \delta_{ml} \\ \pi \ t^2 - (1 - \gamma_{ml}) \ \pi \ \delta_{ml}^2 & if \quad t \geq \delta_{ml}. \end{cases}$$

Using cluster expansion (Croxton 1974), Isham gives results for more general Markov processes and a second order approximation for the two-type point processes and shows how to extend to N-type point processes. The author writes the second order approximation for the K-function in the form

$$\text{(8)} \qquad K_{ml}(t) \approx \begin{cases} (1 + \epsilon d_{ml}) \ \pi \ t^2 & \\ -\epsilon^2 \ d_{ml} \ (g_m \beta_m + g_l \beta_l) \ 2 \ \pi \ J_{ml}(t) + O(\epsilon^3), & t \leq \delta_{ml} \\ \pi \ t^2 + \epsilon \ d_{ml} \ \pi \ \delta_{ml}^2 & \\ -\epsilon^2 \ d_{ml} \ (g_m \beta_m + g_l \beta_l) \ 2 \ \pi \ J_{ml}(t) + O(\epsilon^3), & t \geq \delta_{ml} \end{cases}$$

where $J_{ml}(s) = \int_0^s s\ A_{ml}(s)\ ds$, with $A_{ml}(\parallel \xi - \eta \parallel) = A_{ml}(\xi, \eta)$ being the area of the region in the plane whose points are neighbors of both a type m point in ξ and a type l point in η. That is, $A_{ml}(s)$ is the area of intersection of two discs centered s units apart, both with radius δ_{ml}. Isham is aware that a form of second order approximation for $K_{mm}(t)$ can be written down easily but, more generally, for $K_{ml}(t)$ the problem becomes too complex, since it involves integrating A_{ml}.

2.2. An approximation for a Strauss disc process with continuous mark distribution

Using the same method described in the previous section, we develop here an extension for the Strauss disc process with density (4) to be presented in this section. Consider $\rho = \tau \times \mu$, where τ is Lebesgue measure, yielding the unmarked Poisson process of unit intensity, and μ corresponding to the distribution of the marks. Denote by R, R_1 and R_2 random variables with distribution μ. Following Isham (1984), assume that the edge effects are negligible. Thus

$$\int_{(\Omega\times\Delta)^2} \chi(\parallel x_1 - x_2 \parallel, r_1, r_2)\ \rho(d[x_1, r_1])\ \rho(d[x_2, r_2]) = \pi |\Omega| E_{\mu\times\mu}(R_1 + R_2)^2 \tag{9}$$

and $\gamma = 1 - g\epsilon, \quad g > 0$, for $\epsilon > 0$ small.

For a fixed number of points n, the density in (4) can be written as

$$f(x, r) = z\beta^n \prod_{i<j} \{1 - g\ \epsilon\ \chi(\parallel x_i - x_j \parallel, r_i, r_j)\}$$

which can be approximated by

$$f(x, r) = z\beta^n \{1 - \sum_{i<j} g\ \epsilon\ \chi(\parallel x_i - x_j \parallel, r_i, r_j) + O(\epsilon^2)\}. \tag{10}$$

Let $\sigma^2 = E_{\mu\times\mu}(R_1 + R_2)^2$ and $\rho(d[\mathbf{x}, \mathbf{r}]) = \rho(d[x_1, r_1]) \cdots \rho(d[x_n, r_n])$. A first order approximation for the constant of proportionality is given by

$$
\begin{aligned}
z^{-1} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Omega\times\Delta)^n} f(x,r)\ \rho(d[\mathbf{x},\mathbf{r}]) \\
&\overset{(9)\ (10)}{=} \sum_{n=0}^{\infty} \frac{1}{n!}\ \beta^n\ |\Omega|^n - g\ \epsilon \sum_{n=0}^{\infty} \frac{1}{n!}\ \beta^n\ \frac{n(n-1)}{2}\ |\Omega|^{n-1}\ \pi\ \sigma^2 + O(\epsilon^2) \\
&= e^{|\Omega|\beta} \left(1 - g\ \epsilon\ \pi\ \sigma^2\ \frac{|\Omega|}{2}\ \beta^2 + O(\epsilon^2)\right). \qquad (11)
\end{aligned}
$$

The intensity of a point on ξ with mark r_ξ is given by

$$
\begin{aligned}
\lambda(\xi, r_\xi) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Omega\times\Delta)^n} f(x\cup\xi, r\cup r_\xi)\rho(d[\mathbf{x},\mathbf{r}]) \\
&\overset{(10)}{=} \beta\ z \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \int_{(\Omega\times\Delta)^n} \Bigg\{ 1 - g\ \epsilon \sum_{i<j} \chi(\parallel x_i - x_j \parallel, r_i, r_j) \\
&\quad - g\ \epsilon \sum_{i=1}^{n} \chi(\parallel \xi - x_i \parallel, r_\xi, r_i) + O(\epsilon^2) \Bigg\}\ \rho(d[\mathbf{x},\mathbf{r}]) \\
&\overset{(9)}{=} \beta\ z\ e^{|\Omega|\beta} \Big\{\ 1 - g\ \epsilon\ \pi\ \sigma^2\ \frac{|\Omega|}{2}\ \beta^2 - g\ \epsilon\ \beta\ \pi\ E_\mu(r_\xi + R)^2 + O(\epsilon^2) \Big\} \\
&= \beta \Big\{\ z\ e^{|\Omega|\beta} \Big[\ 1 - g\ \epsilon\ \pi\ \sigma^2\ \frac{|\Omega|}{2}\ \beta^2 \Big]\ \Big[\ 1 - g\ \epsilon\ \beta\ \pi\ E_\mu(r_\xi + R)^2 \Big] \\
&\quad + O(\epsilon^2) \Big\} \\
&\overset{(11)}{=} \beta \Big\{\ 1 - g\ \epsilon\ \pi\ \beta\ E_\mu(r_\xi + R)^2 + O(\epsilon^2) \Big\}\ .
\end{aligned}
$$

The unconditional intensity is then approximated by $\lambda \approx \beta\{1 - g\ \epsilon\ \pi\ \beta\ \sigma^2\}$. Using the same arguments, we find an approximation for the joint intensity, which is given by

$$
\begin{aligned}
\lambda(\xi,\eta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Omega\times\Delta)^n} f(x\cup\xi\cup\eta, r\cup r_\xi\cup r_\eta)\rho(d[\mathbf{x},\mathbf{r}]) \\
&= \beta^2\ z\ e^{|\Omega|\beta} \Big\{\ 1 - g\ \epsilon\ \pi\ \sigma^2\ \frac{|\Omega|}{2}\ \beta^2 - 2\ g\ \epsilon\ \beta\ \pi\ \sigma^2 \\
&\quad -\ g\ \epsilon\ E_{\mu\times\mu}(\chi(\parallel \xi - \eta \parallel, R_1, R_2)) + O(\epsilon^2) \Big\} \\
&= \beta^2\ z \Big\{\ e^{|\Omega|\beta} \Big[1 - g\ \epsilon\ \pi\ \sigma^2\ \frac{|\Omega|}{2}\ \beta^2\Big] \Big[1 - g\ \epsilon\ \beta\ \pi\ \sigma^2\Big]^2 \\
&\quad \Big[1 - g\ \epsilon\ E_{\mu\times\mu}(\chi(\parallel \xi - \eta \parallel, R_1, R_2))\Big] + O(\epsilon^2) \Big\} \\
&\approx \lambda^2 \Big\{\ 1 - g\ \epsilon\ E_{\mu\times\mu}(\chi(\parallel \xi - \eta \parallel, R_1, R_2)) \Big\}\ .
\end{aligned}
$$

Through these approximations we can access the second-order property

$$
\begin{aligned}
K(t) &\approx \int_0^t 2\pi u \left[1 - g\epsilon \int_{\Delta^2} \chi(u, r_\xi, r_\eta)\ \mu(r_\xi)\mu(r_\eta)\ dr_\xi dr_\eta\right]\ du \\
&= \pi t^2 - g\epsilon 2\pi \int_{\Delta^2} \mu(r_\xi)\mu(r_\eta) \left(I[t \le r_\xi + r_\eta] \int_0^t u\ du \right. \\
&\quad \left. + I[t \ge r_\xi + r_\eta] \int_0^{r_\xi + r_\eta} u\ du \right)\ dr_\xi dr_\eta \\
&= \pi t^2 - (1-\gamma)\pi \Big\{\ t^2 P(R_1 + R_2 \ge t) \\
&\quad + E_{\mu\times\mu}\left((R_1+R_2)^2 | R_1 + R_2 \le t\right) \Big\}\ .
\end{aligned}
$$

In this way we have an extension to what Isham developed for a discrete set of types. Notice that we have an approximation for the K-function which depends on the distribution of the sum of pairs of radii. In the multitype case, we can have a set of $N(N+1)/2$ K-functions to estimate.

A second order approximation for the K-function in this case is rather complex, but we can use the definition of the K-function (7) and apply the form obtained by Isham for the multitype point processes (8). In this case, fixing the number of points n, we have that $\gamma_{ml} = \gamma$ and $\beta_m = \beta$ for $m, l = 1, \cdots, N$ with $N \le n$. Then

$$
K_{ml}(t) \approx \gamma\ \pi\ t^2\ +\ 4\ \pi\ (1-\gamma)^2\ \beta\ J_{ml}(t)\ +\ (1-\gamma)\ \pi\ (t^2 - {\delta_{ml}}^2)\ I(t \ge \delta_{ml})
$$

with $I(\cdot)$ being the indicator function. The average reduced second moment measure is

$$
\begin{aligned}
\lambda K(t) &= E_\mu \left\{ \sum_{l=1}^N \lambda_l\ K_{ml}(t) \right\} \\
&\approx \gamma\ \pi\ t^2 \sum_{l=1}^N \lambda_l + 4\ \pi\ (1-\gamma)^2\ \beta \sum_{l=1}^N \lambda_l\ E_\mu\left[J(t, R_1, \delta_{ll})\right] \\
&\quad + (1-\gamma)\ \pi \sum_{l=1}^N \lambda_l\ E_\mu\left[(t^2 - (R_1 + \delta_{ll})^2)\ I(t \ge R_1 + \delta_{ll})\right].
\end{aligned}
$$

An extension to Isham's second order approximation results is given by

$$
\begin{aligned}
K(t) &\approx \gamma\ \pi\ t^2 + 4\ \pi\ (1-\gamma)^2\ \beta\ E_{\mu\times\mu}\left[J(t, R_1, R_2)\right] \\
&\quad + (1-\gamma)\pi \left\{ t^2 P(R_1 + R_2 \le t) - E_{\mu\times\mu}\left[(R_1+R_2)^2 \mid R_1 + R_2 \le t\right] \right\}.
\end{aligned}
$$

We tried to study this second order approximation, calculating $E_{\mu\times\mu}[J(t, R_1, R_2)]$ numerically, which would be in the form of a triple integral. The results (not shown) did not encourage us to continue further. The fact that this approximation includes a term that is proportional to $(1-\gamma)^2\ \beta$ makes the approximation even worse for values of γ that are not extremely close to 1 when β is reasonably big.

3. Simulation study for Strauss disc process

In order to check the effectiveness of the first order approximation for the K-function given in the previous section, we conducted a simulation study. Using the Metropolis-Hastings algorithm unconditional on the number of points (Geyer and Møller 1994), we simulated 100 Strauss disc processes subsampled from a Markov chain to create 95% confidence MCMC envelopes for each case considered. After some evaluation using time series plots, we considered a burn-in period of 20,000 basic steps, since equilibrium is reached within first 5,000 basic steps, depending on the parameter γ and radii used. Although we did not investigate about the optimum space among samples, we allowed 200 steps between samples, in order to get a fairly weak correlation among samples and, perhaps a more important feature, which is having a long enough chain for a not so high computational cost. Also, we have considered just the possibility of adding or removing a marked point, following suggestions given by the authors with respect to the simulation of Strauss processes (parameter p in their algorithm equal to 1). For this study, we considered the cases where the radii are constant, or have uniform distribution, or have gamma distribution. We set up the radii distributions keeping the mean r the same in all cases and considered two levels of variance for the uniform case, for each set of parameters β and γ. Here and everywhere in this research we always used Ripley's estimator for the K-function given by (3).

3.1. Constant radii

For discs with constant radii r, the model becomes simply the Strauss point process with hard-core distance equal to $2r$. Figure 1 shows 95% envelopes and mean for the estimated K-function calculated from an MCMC sample of size 100 and the approximation given by (7) with $\delta_{ml} = 2r$ and $\gamma_{ml} = \gamma$ for small values of γ and different values of r, the radius.

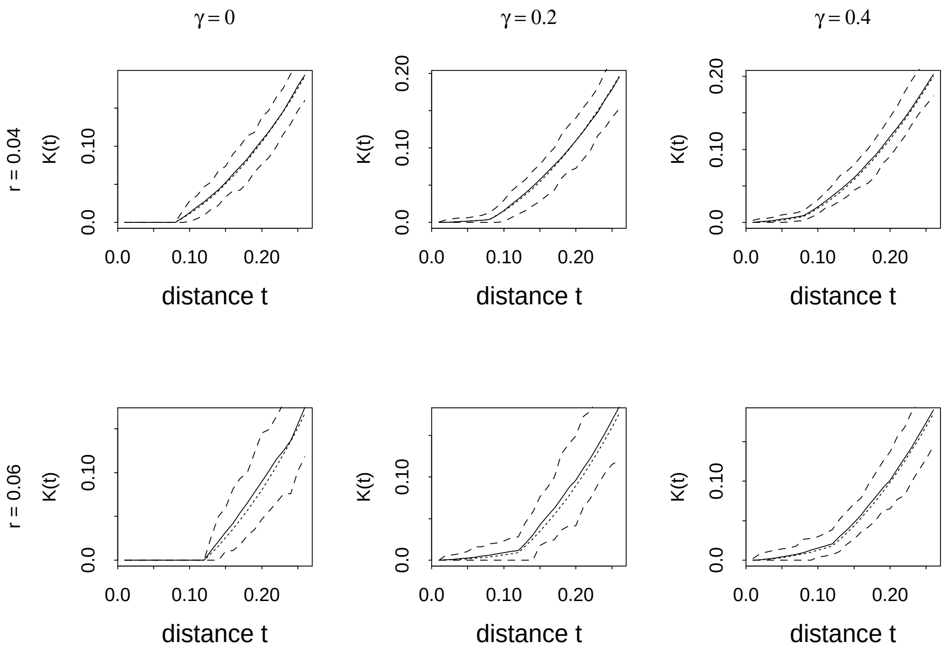

Figure 1. Monte Carlo 95% envelopes for the K-function with mean (solid line) and first order approximation (dashed line) for constant radii (β equal to 50).

We can see that the approximation gets closer to the MCMC mean when we increase γ. On the other hand, when we compare MCMC means of the K-function among different sets of parameters, we notice a decrease in sensitivity of this function as γ gets away from zero, for the same value of r. That is, the difference between $\gamma = 0$

and $\gamma = 0.2$ seems bigger than difference between $\gamma = 0.2$ and $\gamma = 0.4$. Earlier studies actually showed that this difference becomes negligible when we approach the homogeneous Poisson process. Note also that the sensitivity is larger for larger mean radius.

Results show that this approximation is reasonable even for values of γ close to zero for Strauss point processes, which indicates it could be used for estimation of γ or for testing patterns against CSR.

3.2. Radii with Uniform distribution - $U[a,b]$

The K-function approximation for Uniform$[a,b]$ distributed radii is given by

$$K(\mathrm{t}) \approx \begin{cases} \gamma\pi t^2, & t \leq 2a, \\ \pi t^2 - (1-\gamma)\pi \left\{ t^2 \left(1 - \dfrac{(t-2a)^2}{2(b-a)^2} \right) \right. \\ \qquad\qquad \left. + \dfrac{3t^4 - 8at^3 + 16a^4}{6(t-2a)^2} \right\}, & 2a < t \leq a+b, \\ \pi t^2 - (1-\gamma)\pi \left\{ t^2 \left(\dfrac{(2b-t)^2}{2(b-a)^2} \right) \right. \\ \qquad \left. + \dfrac{16a^4 - 2(a+b)^4 + 8bt^3 - 3t^4}{6\{2(b-a)^2 - (2b-t)^2\}} \right\}, & a+b < t \leq 2b, \\ \pi t^2 - (1-\gamma)\pi \left\{ \dfrac{7a^2 + 7b^2 + 10ab}{6} \right\}, & t > 2b. \end{cases}$$

In order to study the behavior of the approximation in this case, we considered distributions with two different variances. Because of the fact that the outcome distribution from simulations in this case are skewed with respect to the "primary" distribution, we investigated how much this fact would affect the approximation. We calculated the approximation from the empirical mark distribution of the generated patterns and we obtained very similar results.

Figure 2 shows when the radii are distributed as uniform in the range $(2r/3, 4r/3)$, with variance equal to $r^2/27$, while in Figure 3, the radii are uniformly distributed on $(r/2, 3r/2)$, with variance equal to $r^2/12$. In both cases the approximation gets away from the MCMC mean as we approach γ to zero (hard core) and also when we increase the radii mean r.

At $t = 2a$, the K-function is not continuous, and the jump at that point is of size

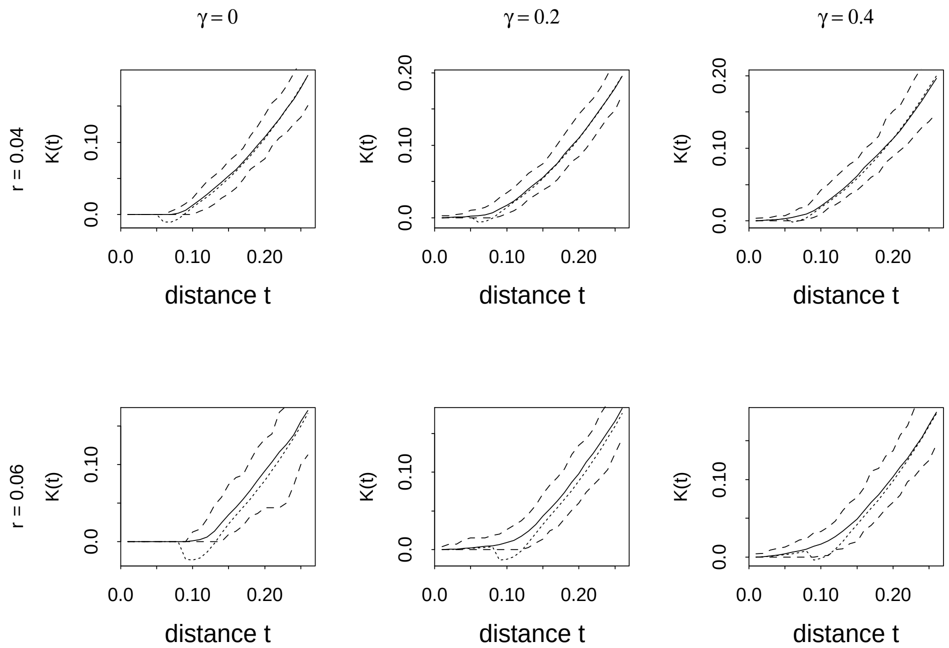

Figure 2. Monte Carlo 95% envelopes for the K-function with mean (solid line) and first order approximation (dashed line) for radii distributed as Uniform[2r/3,4r/3] and β equal to 50.

$4\pi a^2(1-\gamma)$ and so becomes negligible as either a approaches zero or γ approaches one. Another fact we can observe is that the sensitivity of the K-function in detecting changes in parameters of the Strauss disc process becomes weaker than when we have constant radii. As we depart a little from the hard core process, the K-function seems to decrease in sensitivity, giving us an indication that this characteristic itself may not be a good summary in this case of Strauss disc process with continuous radii.

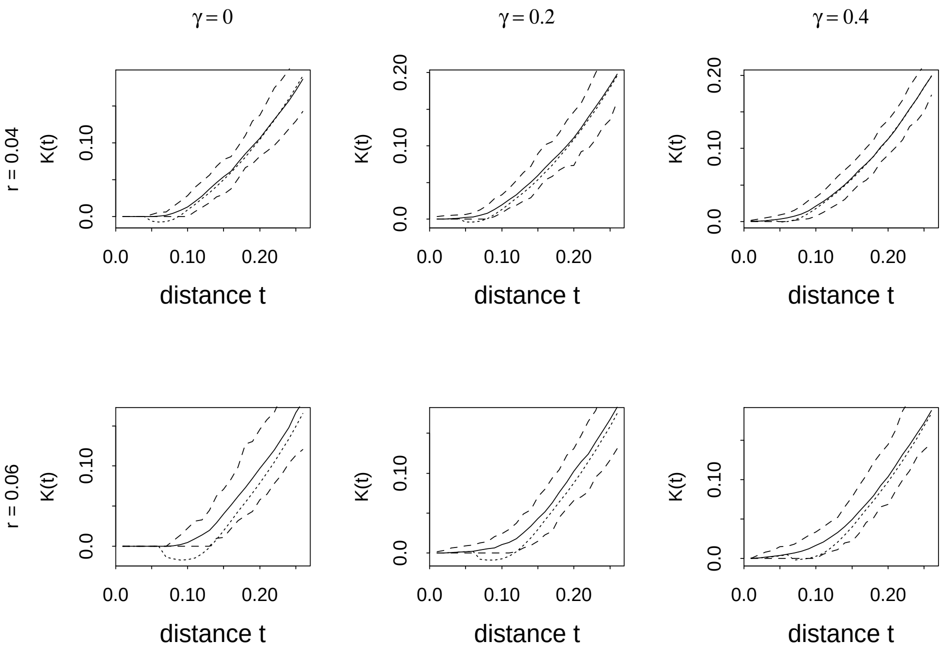

Figure 3. Monte Carlo 95% envelops for the K-function with mean (solid line) and first order approximation (dashed line) for radii distributed as Uniform[r/2,3r/2] and β equal to 50.

3.3. Radii with gamma distribution - $\Gamma(b, c)$

We consider here the case when c is an integer (Erlang distribution) and the density for the radii is given by

$$f_R(u) = \frac{u^{c-1}\ exp(-u/b)}{\Gamma(c)\ b^c}, \quad u > 0, \quad b, c > 0, \quad c \ \text{ integer.}$$

Let's denote the cumulative distribution function of a gamma distributed random variable R by $\Gamma(u, b, c) = P(\Gamma(b, c) \le u)$. Then we have that the approximation for the K-function is given by

$$K(t) \approx \pi t^2 - (1-\gamma)\pi \left\{ t^2\ (1 - \Gamma(t, b, 2c)) + \frac{b^2\ 2\ c(2c+1)\Gamma(t, b, 2c+2)}{\Gamma(t, b, 2c)} \right\}.$$

Figure 4 shows the approximation for the gamma distributed radii. The approxima-

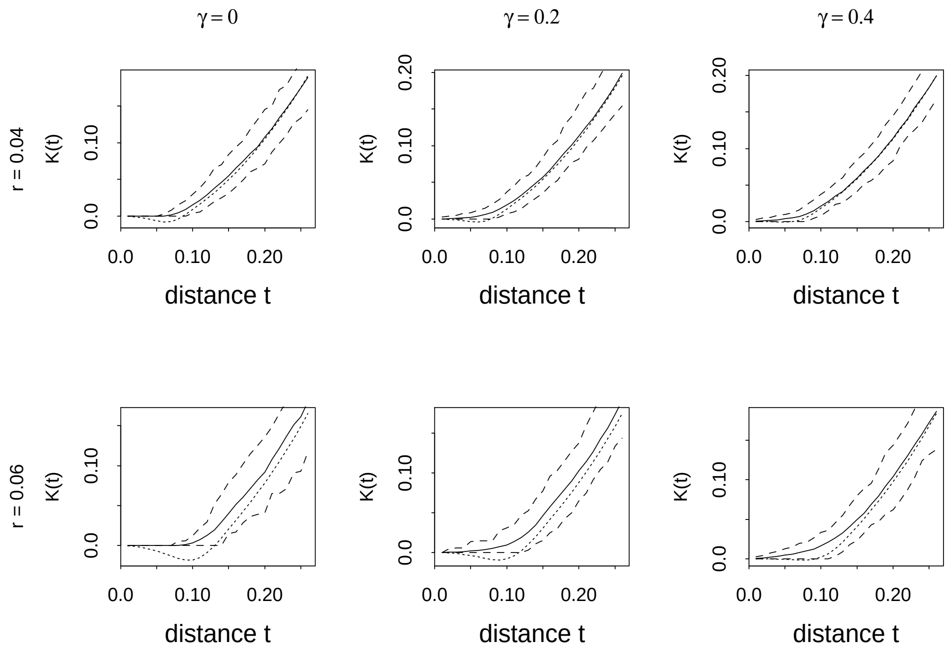

Figure 4. Monte Carlo 95% envelopes for the K-function with mean (solid line) and first order approximation (dashed line) for radii distributed as $\Gamma(r/12, 12)$ and β equal to 50.

tion is not good for values close to zero and, for mean radii 0.04, it starts getting closer to the MCMC estimate of the K-function. We can compare Figures 3 and 4, considering two distributions with the same mean and variance but different shapes. There is no evident difference between the MCMC means and envelopes of the K-function for the two radii distributions, indicating that the K-function itself might not be sensitive to change in shapes of the radii distribution.

4. Analysis and conclusions

The first order approximation for the K-function by this method (cluster expansion) is not efficient, since for values close to zero (where the assumptions on which the method is based fail) there is a substantial departure from the estimated K-function. While for the case of constant radii (Strauss processes) this departure is almost negligible, for the case of Strauss disc processes it becomes bigger as γ approaches to zero.

When we analyze the MCMC means of the K-functions, we see that this function is not sensitive to γ for the same radii distribution. Most probably this is because the continuous distribution of the marks makes the K-function smoother when we allow any level of overlapping of the discs, especially if the range of the disc radii covers values close to zero. Therefore this leds us to question the use of K-function differences as a basis for comparison of Strauss disc type patterns.

	β	25		100	
	r	0.04	0.06	0.02	0.03
γ	0.0	0.170	0.222	0.034	0.045
	0.1	0.158	0.222	0.045	0.042
	0.2	0.192	0.186	0.041	0.040
	0.3	0.153	0.222	0.038	0.045
	0.4	0.177	0.189	0.041	0.039
	0.5	0.128	0.159	0.035	0.039
	0.6	0.148	0.136	0.033	0.035
	0.7	0.144	0.140	0.038	0.028
	0.8	0.138	0.161	0.029	0.037
	0.9	0.127	0.148	0.034	0.039
	1.0	0.141	0.145	0.028	0.043

Table 1. Maximum range in the 95% Monte Carlo envelopes for 100 Strauss disc patterns with radii distributed $\Gamma(r/12, 12)$.

We did a small simulation study using the Strauss disc process with radii gamma distributed to study the behavior of Ripley's K-function estimator for Strauss disc processes with respect to the β-dependent count of the point process, mean radius r and parameter of interaction γ. This estimator is suppose to be unbiased for values of t less than or equal to an appropriate t_0, here 0.25 (used by Ripley (1979) for 25 point patterns). Table 1 shows results for the maximum range of the 95% confidence interval in MCMC simulations of 100 Strauss disc patterns. The variability of the estimator of the K-function seems to decrease considerably when we enlarge the amount of information, if we compare patterns with $\beta = 25$ and mean radius r to the ones with $\beta = 100$ and $r/2$ with the same parameter of interaction γ. On the other hand, when we have $\beta = 25$, as we increase r or decrease γ (leading to a decrease in the number of points in the pattern), the variability of the estimator increases. These facts may indicate that this estimator for the K-function improves as we enlarge either the density of points, or size of the studied region in this case of Strauss disc processes. But, of course, a more rigorous study is needed to show consistency of the estimator.

The first order approximations extended for the Strauss disc process, as presented in this chapter, seem to work better for values of γ not close to zero, and with possible better behavior as the mean radius decreases. On the other hand, evidence suggests that the K-function becomes smoother and closer to the K-function of the homogeneous Poisson process as we increase the variance of the radii for γ not too far from zero. This fact is discouraging in terms of using this approximation for comparison purposes.

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