Review

1. Errors in Hypothesis Testing

	H_0 is true	H_0 is false
Reject H_0	Type I error	–
Accept H_0	–	Type II error

2. Power Calculations

   - α=P(Type I error)=P(reject H_0 when H_0 is true)
   - β=P(Type II error)=P(accept H_0 for a particular alternative)
   - The *power* is the probability of rejecting H_0 given that the true value of the parameter being tested is some specified value.

     power= α when H_0 is true,

     power= $1-\beta$ for a particular alternative.

3. Sample size determination

   - One-sided test : $n = \sigma^2 \times \frac{(Z_\alpha + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_o})^2}$
   - Two-sided test : $n = \sigma^2 \times \frac{(Z_{\frac{\alpha}{2}} + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_o})^2}$
   - For a $(1-\alpha)$ confidence interval: $n = (\frac{2 \times Z_{\alpha/2} \times \sigma}{W})^2$, where W is the width of the interval.

Practice Problem

1. A machine fills milk bottles, the mean amount of milk in each bottle is supposed to be 32 Oz with a standard deviation of 0.06 Oz. Suppose the mean amount of milk is approximately normally distributed. To check if the machine is operating properly, 36 filled bottles will be chosen at random and the mean amount will be determined.

   (a). If an $\alpha = .05$ test is used to decide whether the machine is working properly, what should the rejection criterion be?

   (b). Find the power of the test if the true mean takes on the following values: 31.97, 31.99, 32, 32.01, 32.03. Draw the power curve.

   (c). Find the probability of a type II error when the true mean is 32.03.

2. A certain type of seed has always grown to a mean height of 8.5 inches, with a standard deviation of 1 inch. Based on past experiment, the mean height of a seed is known to be distributed approximately normal. A researcher wishes to find out whether some new enriched conditions would improve the mean height. He wants to use $\alpha = .01$ test and would like to have a 96% chance of rejecting the null hypothesis if the mean height is 9.5 inches.

   (a). Determine the sample size for the test and the rejection criterion.

   (b). Repeat (a) for an $\alpha = 0.025$ test.

   (c). Repeat (a) for an $\alpha = 0.05$ test.

   (d). Repeat (c) for a power of 0.90 at mean height = 9.5.

   (e). Explain how α, power, and sample size are related.

3. A standard insecticide is used to control a particular insect pest in soy beans. The probability with which this insecticide eliminates the insect from an individual plant infested with the insect is 0.7. A new insecticide is developed which is known not to decrease the probability of insect elimination; it is desired to determine if the new insecticide performs better than the standard. An experiment is to be conducted in which 500 randomly selected plants (known to be infested with the insect) are sprayed with the new insecticide. After a fixed period of time, the plant for which the new insecticide eliminate the insect will be counted. The experiment will reject the null hypothesis that the insecticide has no effect if the insecticide eliminates the insect in 375 or more plants.

   (a). State symbolically the null and alternative hypotheses, defining all symbols in words.

   (b). What is the α-level of the proposed test?

   (c). If in fact the success rate of the insecticide is 0.8, what is the power of the test?

   (d). When the experiment was actually carried out, it was found that new insecticide eliminates insect in 395 plants. Find the P-value of the test.

Solution

1. (a) $X \sim N(\mu, \sigma^2)$,$\sigma^2 = 0.06^2$, so $\bar{X} \sim N(\mu, \sigma^2/36)$.

$$H_0 : \mu = 32, \quad H_A : \mu \neq 32,$$

$$\begin{aligned} \alpha = 0.05 &= P(\bar{X} \geq 32 + a) + P(\bar{X} \leq 32 - a) \\ &= 2P(\frac{\bar{X} - 32}{\sigma/6} \geq \frac{a}{\sigma/6}) \\ &= 2P(Z \geq \frac{6a}{\sigma}) \end{aligned}$$

therefore, $\frac{6a}{\sigma} = z_{0.025} = 1,96, a = 0.0196$. Hence the rejection regios are $32 \pm a = 32.0196, 31.9804$.

(b)

$$\begin{aligned} Power_1 &= P(\bar{X} > 32.0196 \mid \mu = 31.97) + P(\bar{X} < 31.9804 \mid \mu = 31.97) \\ &= 0.8508 \\ Power_2 &= P(Z > 2.96) + P(Z < -0.96) = 0.1700 \\ Power_3 &= P(Z > 1.96) + P(Z < -1.96) = 0.05 \\ Power_4 &= P(Z > 0.96) + P(Z < -2.96) = 0.1700 \\ Power_5 &= P(Z > -1.04) + P(Z < -4.96) = 0.8508 \end{aligned}$$

(c)

$$\beta = 1 - Power_5 = 0.1492$$

2. (a) $\mu_{H_0} = 8.5$, $\mu_{H_A} = 9.5$, $\alpha = 0.01$, $1 - \beta = 0.96$, so $\beta = 0.04$, $\sigma^2 = 1$

$$H_0 : \mu = 8.5$$
$$H_A : \mu > 8.5$$

So, $n = \sigma^2 \times \frac{(Z_\alpha + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_0})^2} = 16.64 \approx 17$

(b) $\alpha = 0.025$
$n = \sigma^2 \times \frac{(Z_\alpha + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_0})^2} = 13.76 \approx 14$

(c) $\alpha = 0.05$
$n = \sigma^2 \times \frac{(Z_\alpha + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_0})^2} = 11.49 \approx 12$

(d) $\alpha = 0.05$, $1 - \beta = 0.9$, so $\beta = 0.1$, $\mu_{H_A} = 9.5$
$n = \sigma^2 \times \frac{(Z_\alpha + Z_\beta)^2}{(\mu_{H_A} - \mu_{H_0})^2} = 8.52 \approx 9$

(e) $\alpha \nearrow$ OR $Power \searrow \longrightarrow n \searrow$

3. (a) $X \sim Bi(500, P)$

$$H_0 : \mu = 0.7$$
$$H_A : \mu > 0.7$$

(b)

$$\begin{aligned} \alpha &= P(X \geq 375 \mid H_0) \\ &\approx P((\frac{X - 500 \times 0.7}{\sqrt{(500 \times 0.7 \times 0.3)}}) \geq (\frac{375 - 500 \times 0.7}{\sqrt{(500 \times 0.7 \times 0.3)}}) \mid H_0) \\ &\approx P(Z > 2.4398) \\ &= 0.073 \end{aligned}$$

(c)

$$\begin{aligned} Power &= P(X > 375 \mid C = 0.8) \\ &\approx P(Z \geq \frac{375 - 500 \times 0.8}{\sqrt{(500 \times 0.8 \times 0.2)}}) \\ &= P(Z \geq -2.80) \\ &= 0.9974 \end{aligned}$$

(d)

$$\begin{aligned} P &= P(X \geq 395 \mid H_0) \\ &\approx P(Z \geq \frac{395 - 500 \times 0.7}{\sqrt{(500 \times 0.7 \times 0.3)}}) \\ &= P(Z > 4.39) < 0.0001 \end{aligned}$$