Among the things I learned as a student at Madison are

1) think as Bayesian -- George Box & George Tiao
2) think in function spaces -- Grace Wahba

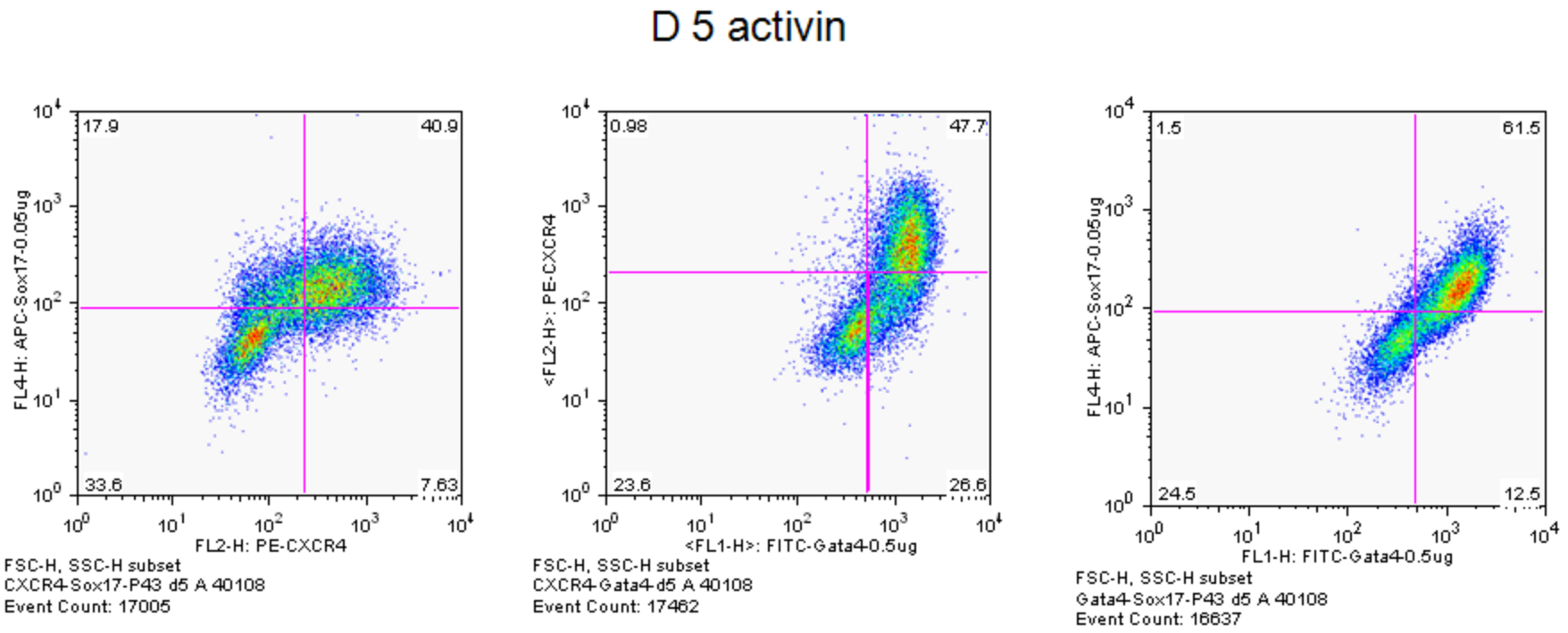
Today's talk will show that I haven't forgotten these lessons

Optional Polya Tree & Bayesian Inference

Wing H. Wong

(Wong & Li, Annals of Statistics, 2010, June issue)

Example 1: modeling flow cytometry data by a density in R^k



Genotype of subject

Disease Status

no

no

no

yes

no

no

Example 2:

Modeling joint status of markers by a 3^k table

The Bayesian nonparametric problem

- $x_1, x_2, \ldots x_n$ are independent r.v. on a space Ω
- drawn from a common distribution Q on Ω
- Q is unknown but assumed to have a prior distribution π.

Our task is to construct a class of priors for the distribution Q so that Bayesian Inference on Q is feasible

Want this to work well in moderate dimensions, e.g. k=10

Ferguson's conditions (1973)

- Support of π should be large
- The corresponding posterior should be tractable

Dirichlet process prior with parameter α:

Q( ) is a stochastic process indexed by subsets of Ω such that for disjoint sets A_i's,

$$(Q(A_i), i=1,\ldots k) \sim \text{Dirichlet } (\alpha(A_i), i=1,\ldots k)$$

where α is a measure on Ω.

Ferguson shows that this prior satisfies the two conditions.

However, when Ω is Euclidean the random distribution Q does not possess a density

Density is needed in many applications

- Estimating Kullback-Leibler divergence

  $\int \log(g/f)\, f(x)dx$

- Error distribution for regression

  $y_i = g(x_i\,;\,\theta) + \varepsilon_i$ where ε has density $q(\ )$

  Likelihood $(\theta \mid q) = \prod q(\, y_i - g(x_i\,;\,\theta)\,)$

Partitioning scheme

Suppose Ω can be partitioned in one of M ways:

For j= 1, 2, …, M,

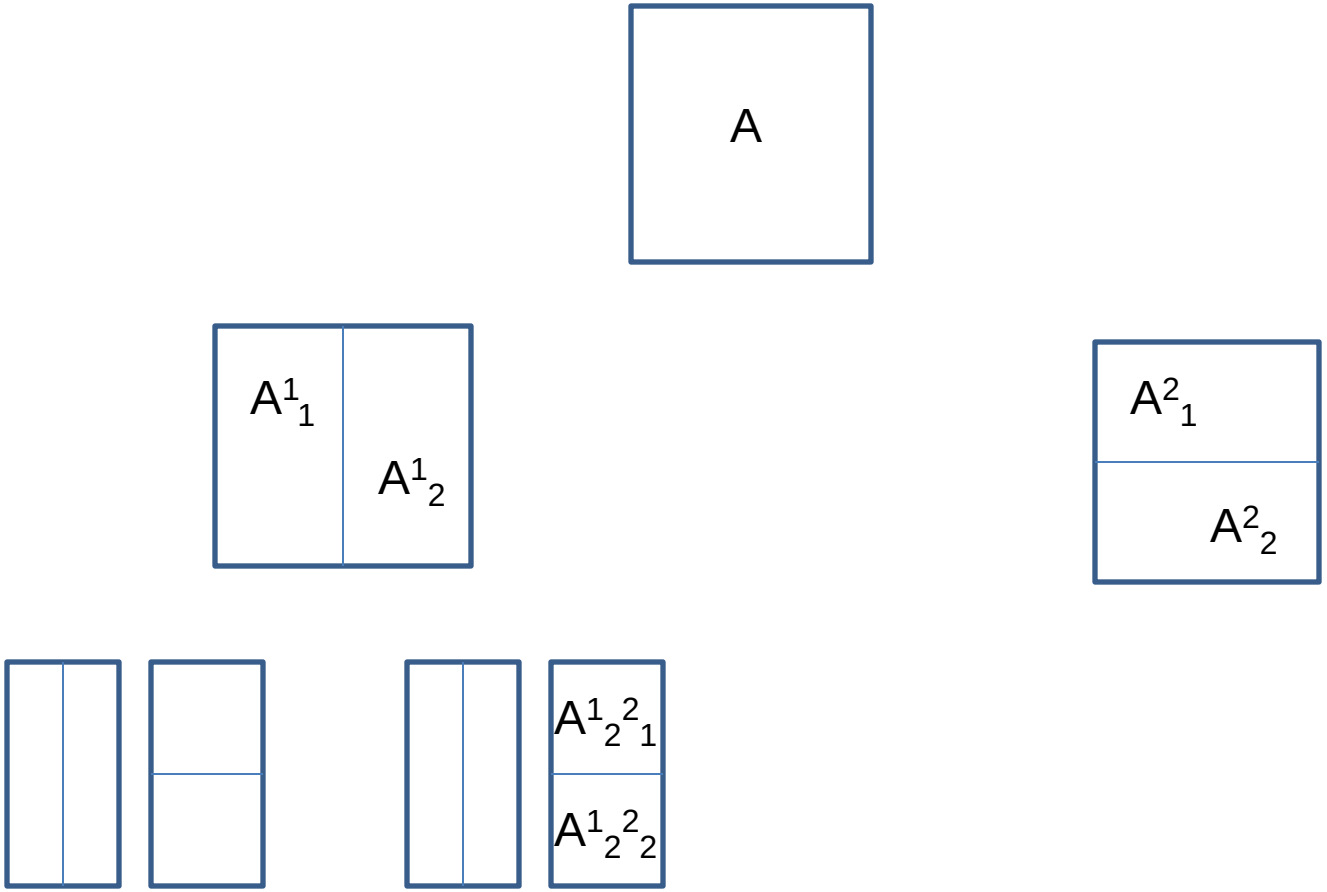
$$\Omega = \bigcup_{k=1}^{K^j} \Omega_k^j \qquad \text{where } \Omega_k^j\text{'s are disjoint}$$

Each level-1 elementary region Ω_k^j can be further partitioned in one of several ways into level-2 elementary regions:

$$\Omega_{k_1}^{j_1} = \bigcup_{k_2=1}^{K_{k_1}^{j_1 j_2}} \Omega_{k_1 k_2}^{j_1 j_2}$$

Let $\mathcal{A}^k$ be the set of elementary regions with level = k ,

$\mathcal{A}^{(k)}$ be the set of elementary regions with level ≤ k



In general, A^j_k = k^{th} part of the j^{th} way to partition A

Some Examples

EXAMPLE 1.

$$\Omega = \{x = (x_1, \ldots, x_p) : x_i \in \{1, 2\}\}$$
$$\Omega_k^j = \{x : x_j = k\}, \qquad k = 1 \text{ or } 2$$
$$\Omega_{k_1 k_2}^{j_1 j_2} = \{x : x_{j_1} = k_1, x_{j_2} = k_2\}, \text{ etc.}$$

In this example, the number of ways to partition a level-k elementary region decreases as k increases.

EXAMPLE 2.

$$\Omega = \{(x_1, x_2, \ldots, x_p) : x_i \in [0, 1]\} \subset \mathbb{R}^p$$

If A is a level-k elementary region (a rectangle) and $m_j(A)$ is the midpoint of the range of x_j for A, we set $A_1^j = \{x \in A : x_j \leq m_j(A)\}$ and $A_2^j = A \backslash A_1^j$. There are exactly p ways to partition each A, regardless of its level.

Piecewise constant density

- S ←Ber (ρ), if S=1, $Q^{(1)}$ ← uniform on Ω, stop.
- Else,

  draw J=j with probability=λ_j

  use the j^{th} partition of Ω, i.e., $\Omega = \bigcup_{k=1}^{K^j} \Omega_k^j$

- $\boldsymbol{\theta}^j = (\theta_1^j, \ldots, \theta_K^j)$ ← Dirichlet $(\alpha_1^j, \ldots, \alpha_K^j)$
- $Q^{(1)}$ (Ω_k^j) ← θ^j_k
- $Q^{(1)}$ (| Ω_k^j) ← uniform

Piecewise constant density on partitions of finite depth

- Suppose we have drawn $Q^{(k)}$ supported on a partition composing of regions from $A^{(k)}$
- For each region not yet stopped, repeat the partitioning process
- This gives a random distribution $Q^{(k+1)}$ with a density $q^{(k+1)}$ that is piecewise constant on a partition with regions from $A^{(k+1)}$
- Note: this is just a random recursive partitioning process

Definition of Optional Polya Tree (OPT)

THEOREM 1. *Suppose there is a* $\delta > 0$ *such that with probability* 1, $\rho(A) > \delta$ *for any region* A *generated during any step in the recursive partitioning process. Then with probability* 1, $Q^{(k)}$ *converges in variational distance to a probability measure* Q *that is absolutely continuous with respect to* μ.

i.e. $P\{ \int |q^{(k)} - q| dx \to 0$ *for some density* $q \ \} = 1$

This random probability measure Q is said to have an Optional Polya Tree distribution with parameters $\boldsymbol{\rho}$ *(stopping rule),* $\boldsymbol{\lambda}$ *(selection probabilities) and* $\boldsymbol{\alpha}$ *(probability assignment weights).*

OPT prior has large support in L_1

THEOREM 2. *Let Ω be a bounded rectangle in $\mathbb{R}^p$. Suppose that the condition of Theorem 1 holds and that the selection probabilities $\lambda_i(A)$, the assignment probabilities $\alpha_i^j(A)/(\sum_l \alpha_l^j(A))$ for all i, j and $A \in \mathcal{A}^{(\infty)}$, are uniformly bounded away from 0 and 1. Let $q = dQ/d\mu$, then for any density f and any $\tau > 0$, we have*

$$P\left(\int |q(x) - f(x)| d\mu < \tau\right) > 0$$

Remark: *A useful choice for* $\boldsymbol{\alpha}$ *is*

$$\alpha_i^j(A) = \mu\left(A_i^j\right) \Big/ \mu(A) \qquad \text{for } A \in \mathcal{A}^k$$

OPT prior is conjugate

Theorem 3:

The posterior distribution $\pi(Q \mid x_1, ..x_n)$ *is also OPT with*

1. *Stopping probability:*

$$\rho(A|\boldsymbol{x}) = \rho(A)\Phi_0(A)/\Phi(A)$$

2. *Selection probabilities:*

$$P(J = j|\boldsymbol{x}) \propto \lambda_j \frac{D(\boldsymbol{n}^j + \boldsymbol{\alpha}^j)}{D(\boldsymbol{\alpha}^j)} \prod_{i=1}^{K^j} \Phi\left(A_i^j\right) \qquad j = 1, \ldots, M$$

3. *Allocation of probability to subregions: the probabilities* θ_i^j *for subregion* A_i^j, $i = 1, \ldots, K^j$ *are drawn from Dirichlet* $(\boldsymbol{n}^j + \boldsymbol{\alpha}^j)$.

where

$$\Phi(A) = \int q\left(\boldsymbol{x}(A)|A\right)\, d\pi_A(q)$$

$$\Phi_0(A) = u\left(\boldsymbol{x}(A)|A\right)$$

Computation of Φ(A) by recursion

If $A = \bigcup_{i=1}^{K^j} A_i^j$

then

$$\Phi(A) = \rho\Phi_0(A) + (1-\rho)\sum_{j=1}^{M} \lambda_j \frac{D(\boldsymbol{n}^j + \boldsymbol{\alpha}^j)}{D(\boldsymbol{\alpha}^j)} \prod_{i=1}^{K^j} \Phi\left(A_i^j\right)$$

where $D(\boldsymbol{t}) = \Gamma(t_1)\ldots\Gamma(t_k)/\Gamma(t_1 + \cdots + t_k)$

Termination rule for Recursion (case of 2^p table)

1. A contains no observation. In this case, $\Phi(A) = 1$.
2. A is a single cell (in the 2^p table) containing any number of observations. In this case, $\Phi(A) = 1$.
3. A contains exactly one observation and A is a region where M of the p variables are still available for splitting. In this case,

$$\Phi(A) = \quad 2^{-M}$$

Similarly, termination rules exist for the continuous case

Thus, Ferguson's second condition is also satisfied.

OPT prior leads to asymptotically consistent inference

THEOREM 4. *Let $x_1, x_2, \ldots$ be independent, identically distributed variables from a probability measure Q, $\pi(\cdot)$ and $\pi(\cdot|x_1, \ldots, x_n)$ be the prior and posterior distributions for Q as defined in Theorem 3. Then, for any Q_0 with a bounded density, it holds with $Q_0^{(\infty)}$ probability equal to 1 that*

$$\pi(U|x_1, \ldots, x_n) \longrightarrow 1$$

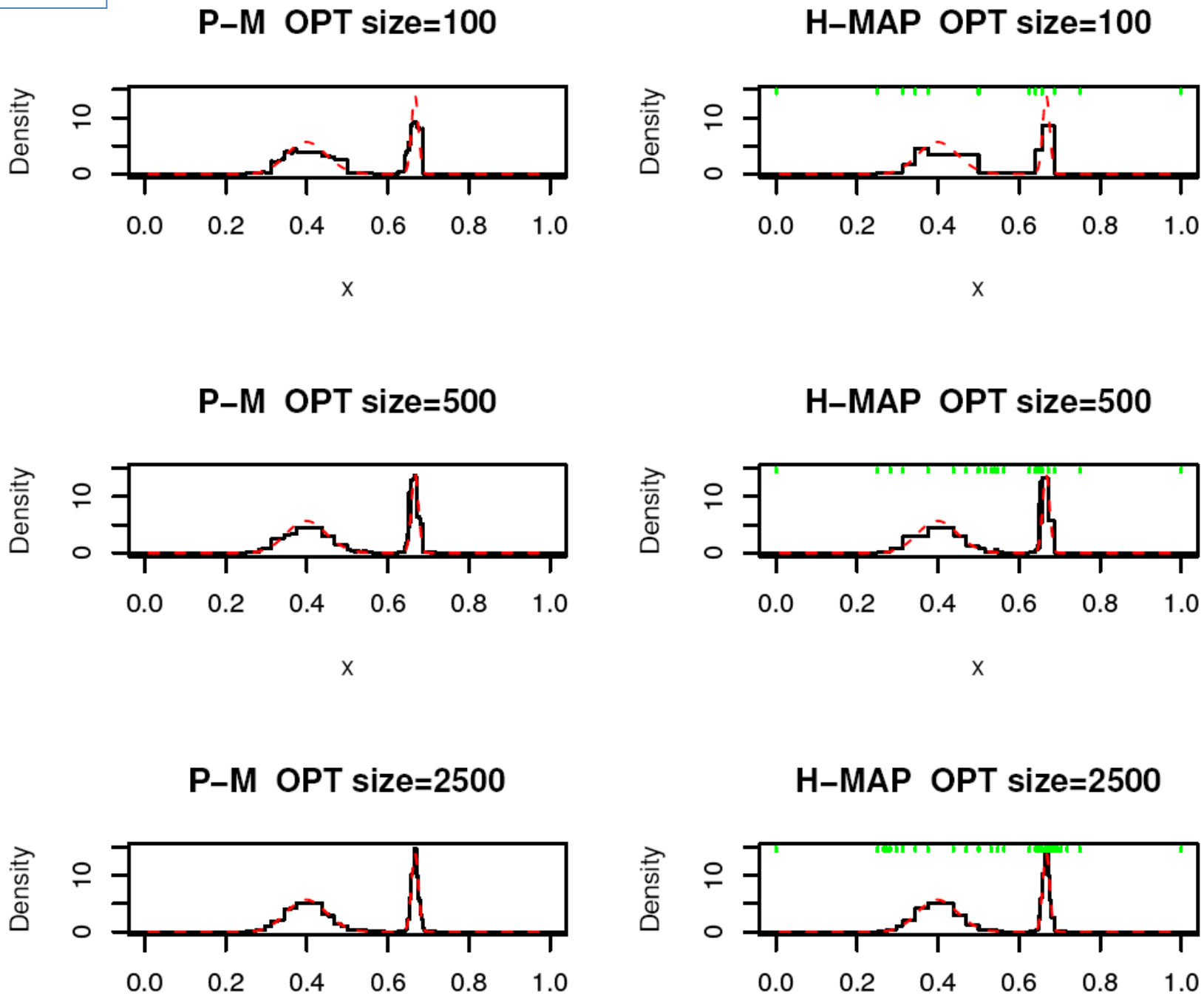
for all weak neighborhoods U of Q_0.

Remark 1: $$U = \left\{ Q : \left| \int g_i(\cdot)\, dQ - \int g_i(\cdot)\, dQ_0 \right| < \epsilon_i, \quad i = 1, 2, \ldots, K \right\}$$

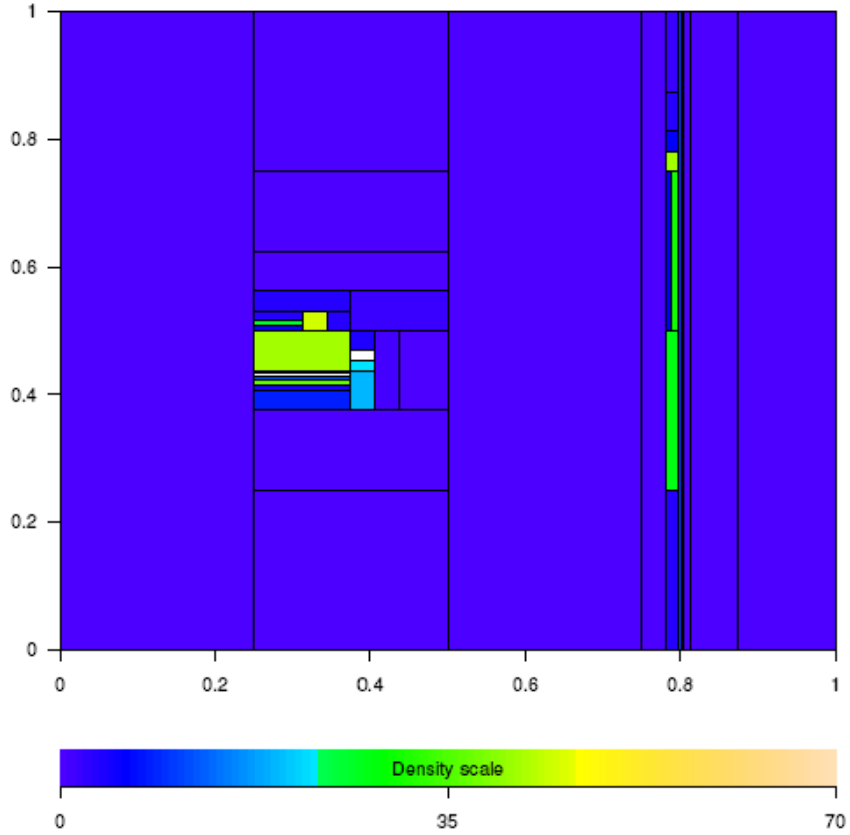
where $g_i(\cdot)$ is a bounded continuous function on Ω.

Remark 2: It should be possible to get rates in Hellinger distance

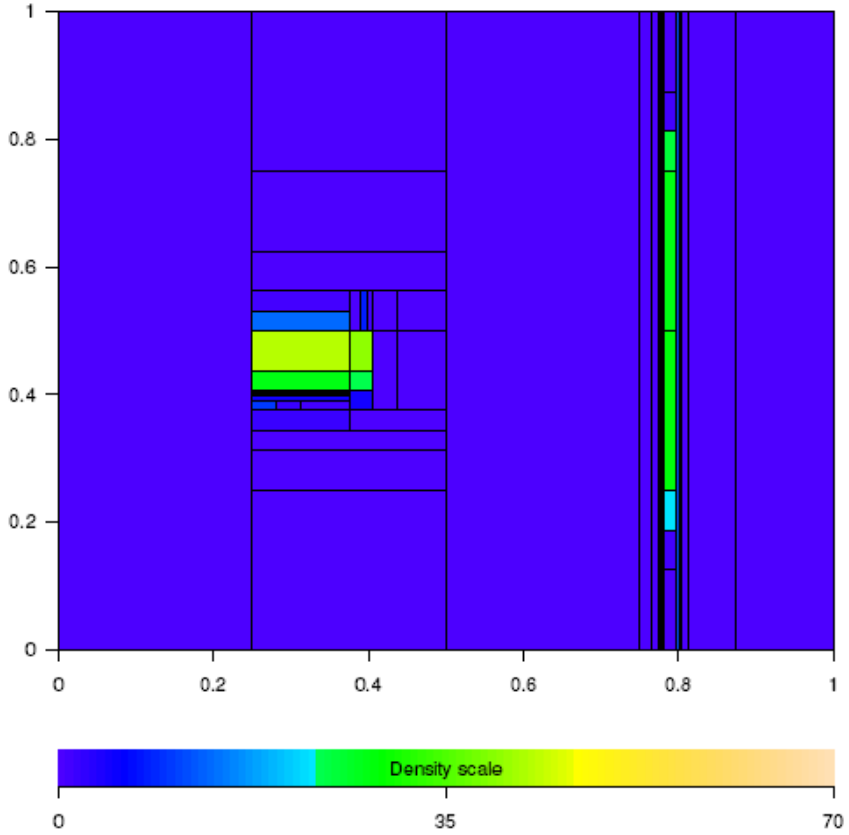
Example 1


P–M OPT size=100
H–MAP OPT size=100
P–M OPT size=500
H–MAP OPT size=500
P–M OPT size=2500
H–MAP OPT size=2500
Density
0
10
0.0
0.2
0.4
0.6
0.8
1.0
x

Example 2

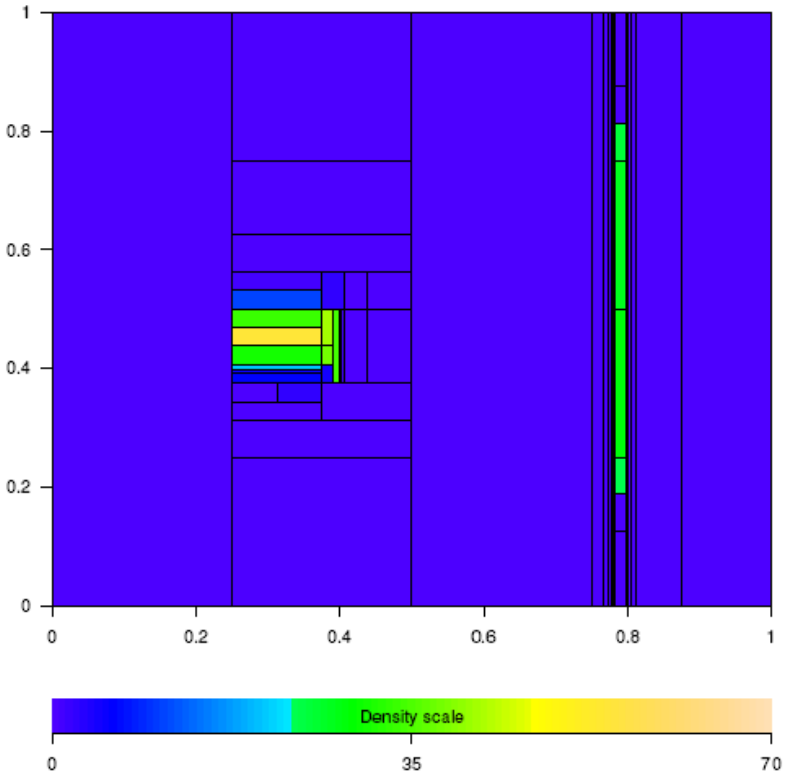


(a) Sample size = 100

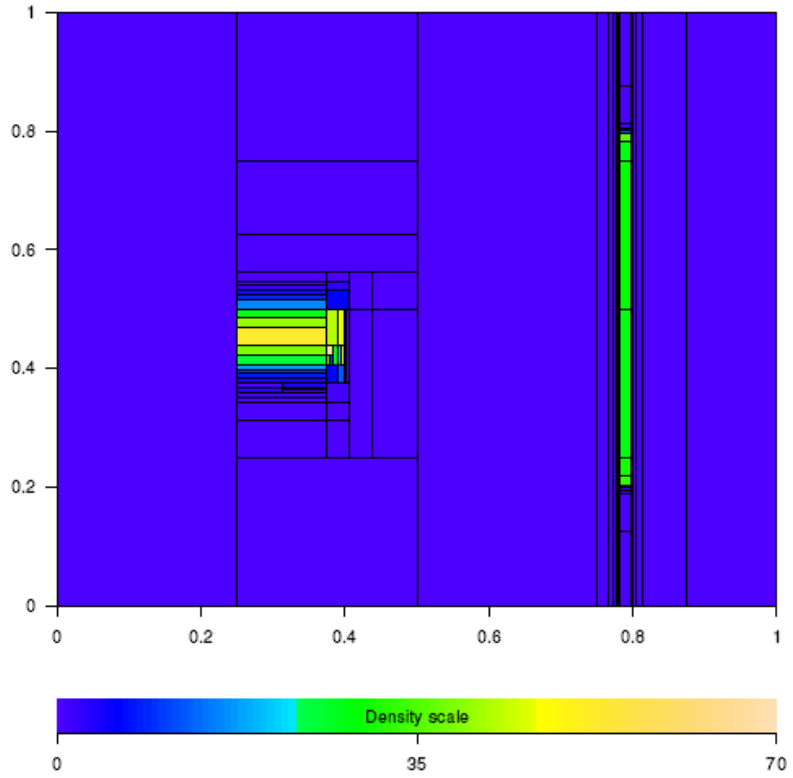


(b) Sample size = 500

Example 2 (continued)



(c) Sample size = 1000



(d) Sample size = 5000

Comparison of two samples

x_0 is a sample from distribution Q_0

x_1 is a sample from distribution Q_1

The OPT can be used to derive tests statistics for the equality of the two distributions.

This is not an easy problem in the multivariate case

One approach based on OPT

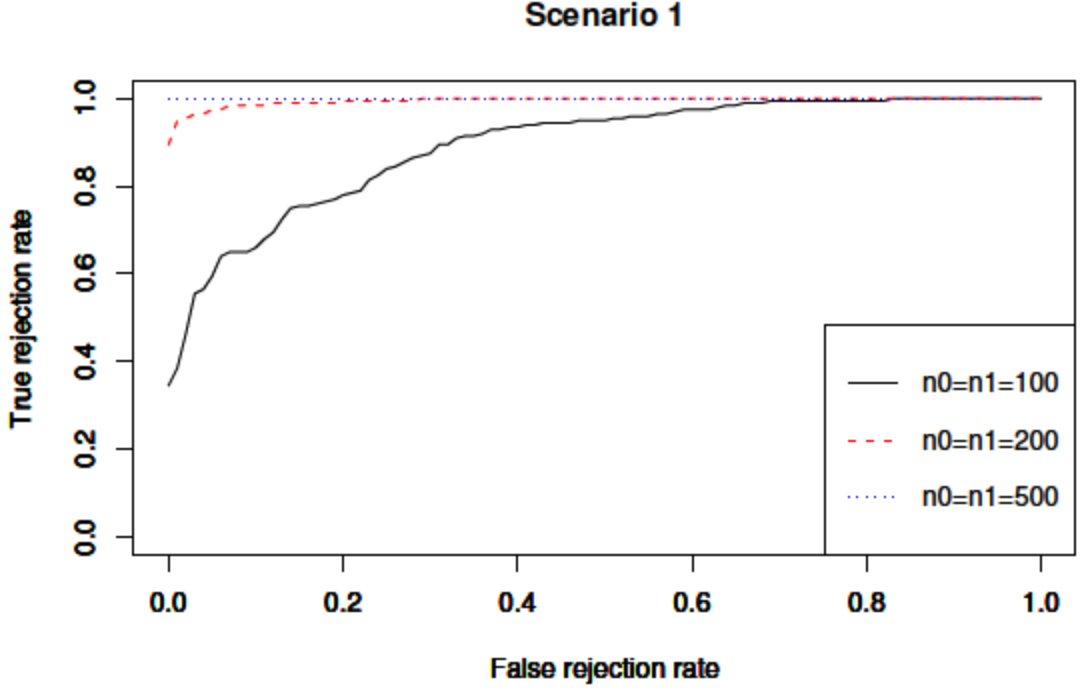
- Given x_0, Q_0 has an OPT as it posterior
- We want to learn a partition for Q_1 that tells us on which parts of the sample space is Q_1 different from Q_0
- When deciding whether to stop or continue to divide A, replace $\Phi_0(A) = u(x_1(A) \mid A)$ by $\Phi_0(A \mid x_0) = \int q_0(x_1(A) \mid A)\, \pi_A(dq_0 \mid x_0)$
- This can be computed by repeating the basic OPT posterior computation twice

Two simulations of case-control samples

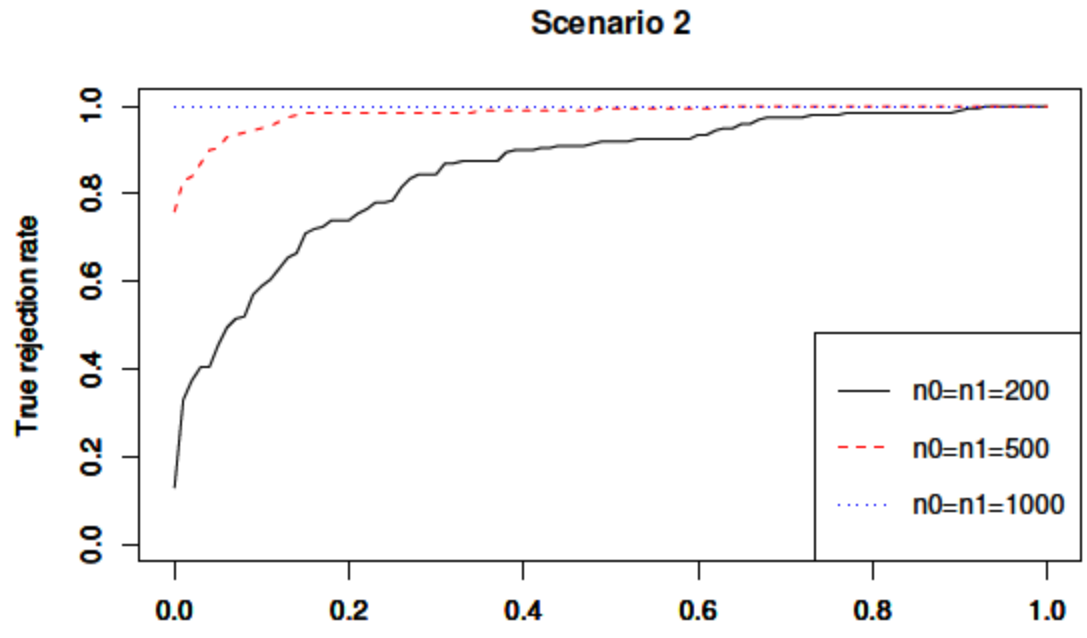
1. $X_1, X_2, \ldots, X_{15} \sim_{i.i.d.} \text{Bernoulli}(0.5)$

2. $X_1, X_2, \ldots X_8$ as a Markov Chain with $X_1 \sim \text{Bernoulli}(0.5)$, and $P(X_t = X_{t-1}|X_{t-1}) = 0.7$, while $X_9, X_{10}, \ldots X_{15} \sim_{i.i.d}$ Bernoulli(0.5) and are independent of $X_1, \ldots, X_8$.

$$Y \sim \begin{cases} Bernoulli(0.3) & \text{if } X_3 = 1 \text{ and } X_7 = 1 \\ Bernoulli(0.2) & \text{if } X_7 = 0 \text{ and } X_{10} = 0 \\ Bernoulli(0.1) & \text{otherwise} \end{cases}$$



For more examples, go to Li Ma's oral, May 19, 2010



An example of partition learned from data