Fractal Function Estimation via Wavelet Shrinkage

Yazhen Wang

University of Missouri-Columbia

Submitted in June 1995

Journal of Royal Statistical Society B 59 (1997), pp. 603-613

Abstract

In scientific studies objects are often very rough. Mathematically these rough objects are modeled by fractal functions, and fractal dimension is usually used to measure their roughness. The present paper investigates fractal function estimation by wavelet shrinkage. It is shown that wavelet shrinkage can estimate fractal functions with their fractal dimensions virtually preserved.

Keywords: FRACTAL DIMENSION; FRACTAL FUNCTION; ROUGH FUNCTION; WAVELETS.

Running Title: Fractal Function Estimation.

1 Introduction

In scientific studies objects are often very far from smooth or piecewise smooth. Examples include fractal signals and images, stock prices, coastlines, and profiles of rough surfaces (Barnsley 1993; Chorafas 1994; Constantine and Hall 1994; Farge, Hunt and Vassilicos 1993; Mandelbrot 1983; Taylor and Taylor 1991; Tricot 1995; Wornell and Oppenheim 1990 1992). These rough objects have many nondifferentiable points, and their plots seemingly look very 'noisy'. Mathematically, fractal functions are used to model these objects, and the fractal dimension (box dimension) is defined to quantitatively characterize these functions and measure their roughness. Generally, the rougher a function is, the larger is its fractal dimension. Algorithms are available to compute fractal dimension by a finite amount of data.

As data collected in practice are usually corrupted with noise, depending on noise contamination and measuring methods, noise often substantially inflates or deflates estimation of fractal dimension (see section 5; Hall 1995; Grassberger et. al. 1993; Kostelich and Yorke 1988). Hence we need to remove noise from a fractal function without considerably altering its fractal dimension. This paper investigates fractal function estimation by wavelet shrinkage for data with additive noise.

Nonparametric function estimation has experienced its fast development in the last one and one-half decades and continues to grow rapidly. A number of smoothing methods including various kernel methods, splines and Fourier methods have been developed. Recently the wavelet shrinkage method has been proposed to estimate functions, in particular rough functions. Wavelet shrinkage is 'denoising' rather than 'smoothing', where denoising generally refers to an attempt to remove noise from a function, and smoothing often implies dampening or downweighting high frequencies on the assumption that they contain more noise than low frequencies (Donoho, Johnstone, Kerkyacharian and Picard 1995). The wavelet estimate has been shown to enjoy spatial adaptivity and to be nearly minimax over a wide range of function classes (Donoho 1993b; Donoho and Johnstone 1994 1995 1996; Donoho, Johnstone, Kerkyacharian and Picard 1995; Wang 1996). For the problem of estimating a fractal function it is very critical to preserve its fractal dimension. Smooth or piecewise smooth functions can't well approximate a fractal function, and normal smoothing of noisy data will substantially reduce its fractal dimension and degrade its quality, so smoothing methods may be inappropriate for fractal function estimation. However, this paper shows that wavelet shrinkage can recover a fractal function from noisy data and virtually preserve its fractal dimension.

Since the concept of fractal sets was introduced and popularized by Benoit Mandelbrot, fractals have now become a fascinating and rapidly growing field, with many new and exciting applications (Barnsley 1993; Mandelbrot 1983; Massopust 1994; Peitgen, Jurgens and Saupe 1992; Tricot 1995). Recently the statistical aspects of fractals have come under explicit study. A whole issue of the Proceedings of the Royal Society, Series A, volume 423 (1989), was devoted to 'Fractals in the Natural Sciences'; seven papers with discussion presented at a Royal Statistical Society Meeting on 'Chaos' were published in the issue of Royal Statistical Society, Series B, volume 54 (1992); and two papers with discussion in

Statistical Science, volume 7 (1992), reviewed 'Chaos, Fractals and Statistics'. Estimates of fractal dimension based on noisy data were investigated by several authors (Constantine and Hall 1994; Hall 1995; Hall and Wood 1993; Ogata and Katsura 1991; Taylor and Taylor 1991).

The paper is organized as follows. Sections 2 and 3 review fractal dimension and wavelets, respectively. The main result is presented in Section 4. Two simulated examples are reported in Section 5, and remarks are given in Section 6.

2 Fractal Dimension

Let $\Gamma = \{(x, f(x)); x \in [0,1]\} \subset I\!R^2$ be the graph of a function $f(x)$ on $[0,1]$. For $\epsilon > 0$, denote by $B(z,\epsilon)$ the disk centered at $z \in I\!R^2$ with radius ϵ. Let $N_\epsilon(\Gamma)$ be the minimum number of disks $B(z,\epsilon)$ that cover Γ, and $L_\epsilon(\Gamma)$ the Lebesgue measure of the ϵ-Minkowski sausage $\Gamma(\epsilon)$ defined by

$$\Gamma(\epsilon) = \bigcup_{z\in\Gamma} B(z,\epsilon).$$

Then the fractal dimension of $f(x)$ is defined as

$$\Delta(f) = \lim_{\epsilon\to 0} \frac{log N_\epsilon(\Gamma)}{|log\epsilon|} = \lim_{\epsilon\to 0}\left(2 - \frac{log L\{\Gamma(\epsilon)\}}{log\epsilon}\right). \tag{1}$$

(See Barnsley 1993; Mandelbrot 1983; Massopust 1994; Taylor and Taylor 1991; Tricot 1995.) We call $f(x)$ a fractal function if $\Delta(f) > 1$.

As $N_\epsilon(\Gamma)$ and $L\{\Gamma(\epsilon)\}$ are usually difficult to analyze directly, we often use the variation of $f(x)$ to find its fractal dimension. Define the ϵ-oscillation of $f(x)$,

$$osc_\epsilon(x) = sup\left\{|f(x_1) - f(x_2)|; x_1, x_2 \in [x-\epsilon, x+\epsilon] \cap [0,1]\right\},$$

and the ϵ-variation

$$V(\epsilon) = \int_0^1 osc_\epsilon(x)dx.$$

Then

$$\Delta(f) = \lim_{\epsilon \to 0}\left(2 - \frac{log\{V(\epsilon)\}}{log\epsilon}\right). \tag{2}$$

(See Massopust 1994, chapter 3; Tricot 1995, chapter 12.) If the limits in (1) and (2) don't exist, we can define lower and upper fractal dimensions by replacing the limits with lower and upper limits, and study lower and upper fractal dimensions (Massopust 1994; Tricot 1995). In this paper we shall not pursue this.

By (2) we can easily show that the Hölderian condition with exponent α,

$$|f(x_1) - f(x_2)| \le C\,|x_1 - x_2|^\alpha, \qquad \text{with } C \text{ a positive constant}, \tag{3}$$

implies $\Delta(f) \le 2 - \alpha$. On the other hand, if there exist an integer $p \ge 1$ and a positive constant C such that for small $h > 0$,

$$\left(\int_h^{1-h} |f(x+h) - f(x-h)|^p\,dx\right)^{1/p} \ge C\,h^\alpha, \tag{4}$$

then $\Delta(f) \ge 2-\alpha$ (Massopust 1994; Tricot 1995, p. 158). Conditions (3) and (4) are often used to relate $\Delta(f)$ to Hölderian exponent α and ensure that $f(x)$ has fractal dimension $2 - \alpha$.

The local dimension of f at a point x_0 is defined as

$$\Delta(f, x_0) = \lim_{\epsilon \to 0} \Delta\{\Gamma \cap B(x_0, \epsilon)\}.$$

While smooth curves, piecewise smooth curves, and curves with finite length all have fractal dimension one, many curves in scientific studies have fractal dimensions greater than one, often with varying local dimensions. These fractal curves are often constructed by self-similarity, self-affinity or iterated function systems (Adler 1981; Barnsley 1993; Mandelbrot 1983; Massopust 1994; Peitgen, Jurgens and Saupe 1992; Tricot 1995; Wornell and Oppenheim 1990 1992).

Given a finite data set, several algorithms are available for computing its fractal dimension (Taylor and Taylor 1991; Tricot 1995, pp. 170-174). The following one will be used in this paper. Let ϵ be the precision of the measurement of a curve and define $Q(\epsilon) = V(\epsilon)/\epsilon^2$, with $V(\epsilon)$ the variation of the curve. Taking decreasing values of ϵ_i, $i = 1, \cdots, m$, we compute $\{|log\, \epsilon_i|, log\, \{Q(\epsilon_i)\}\}$ and run regression on the computed values. The fractal dimension is then estimated by the regression slope.

3 Wavelets

Let ψ be a Daubechies wavelet of compact support (Daubechies 1992; Meyer 1992). Wavelets are derived from ψ by a process of dyadic dilations and translations of ψ. Denote by $\mathbb{Z}$ the set of all integers, and for $j, k \in \mathbb{Z}$ define

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k),$$

where j and k are indexes of resolution level with scale 2^{-j} and spatial position at $k\,2^{-j}$, respectively. Daubechies' construction guarantees that $(\psi_{j,k})_{(j,k)\in\mathbb{Z}^2}$ is a complete orthonormal basis for $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, we have the wavelet decomposition,

$$f(x) = \sum_{j,k} \theta_{j,k}\psi_{j,k}(x), \tag{5}$$

with wavelet coefficient

$$\theta_{j,k} = \int f(x)\psi_{j,k}(x)dx. \tag{6}$$

For functions supported in $[0,1]$, we can select an index set $\Lambda = \{(j,k) : j = 0,1,\cdots,k = 0,\cdots,2^j-1\}$ and modify some of $\psi_{j,k}$, $(j,k)\in\Lambda$, such that $(\psi_{j,k})_{(j,k)\in\Lambda}$ forms a complete orthonormal basis for $L^2[0,1]$ (Cohen, Daubechies, Jawerth, Vial 1993; Daubechies 1994; Donoho and Johnstone 1996).

Wavelets allow us to estimate the smoothness of $f(x)$ in terms of decay estimates of its wavelet coefficients for large j. For example, $f(x)$ satisfies the Hölderian condition with exponent α if, and only, if for some positive constant C,

$$|\theta_{j,k}| \le C\,2^{-j\,(1/2+\alpha)}. \tag{7}$$

(See Daubechies 1992, p. 299; Jaffard 1994, p. 469; Wang 1995.)

Like variation, the wavelet coefficients of $f(x)$ can also be used to compute its fractal dimension as follows,

$$\Delta(f) = \frac{3}{2} + \lim_{j\to\infty}\frac{log_2\left(\sum_k |\theta_{j,k}|\right)}{j}. \tag{8}$$

(See Massopust 1994, corollary 6.2, p. 233.)

By (7) we easily show that Hölderian condition (3) implies that the upper limit of $log_2\left(\sum_k |\theta_{j,k}|\right)/j$ is bounded above by $1/2-\alpha$, and on the other hand, condition (4) infers that the lower limit of $log_2\left(\sum_k |\theta_{j,k}|\right)/j$ is bounded below by $1/2-\alpha$, so (3) and (4) together are sufficient conditions for (8) (Massopust 1994, chapter 6; Tricot 1995, chapter 12). In particular, (8) is satisfied if there exist positive constants C_1 and C_2 such that $C_1 |x_1 - x_2|^\alpha \le |f(x_1) - f(x_2)| \le C_2 |x_1 - x_2|^\alpha$ for all x_1 and x_2.

4 Fractal Dimension of Wavelet Estimate

Suppose data $y_1, \cdots, y_n$ are observed from the nonparametric regression model

$$y_i = f(x_i) + \varepsilon_i, \qquad i = 1, \ldots, n = 2^{J+1}, \tag{9}$$

where $x_i = i/n \in [0,1]$, ε_i are i.i.d random errors with $\varepsilon_i \sim N(0, \sigma^2)$, and $f(x)$ is an unknown function on $[0,1]$ to be estimated.

We use the recently developed wavelet shrinkage method to construct the wavelet estimate $\hat{f}$ of f as follows.

(1). Apply the pyramidal filtering algorithm (Cohen, Daubechies, Jawerth and Vial 1992) to the data (y_i) to compute empirical wavelet coefficients $y_{j,k}$, $j = 0, \cdots, J$, $k = 0, \cdots, 2^j - 1$.

(2). Threshold the empirical wavelet coefficients $(y_{j,k})$ by the soft-threshold rule $\delta_{t_n}(y_{j,k}) = sign(y_{j,k})(|y_{j,k}| - t_n)_+$ or the hard-threshold rule $\delta_{t_n}(y_{j,k}) = y_{j,k}\, 1_{\{|y_{j,k}| \ge t_n\}}$, with threshold t_n. One simple threshold is the universal threshold $t_n = \sigma\, \sqrt{2\, log(n)/n}$. In practice, σ is estimated by the median absolute deviation of the wavelet coefficients $y_{j,k}$ at the finest level J, divided by 0.6745.

(3). Invert the pyramidal filtering and reconstruct the thresholded empirical wavelet coefficients $\delta_{t_n}(y_{j,k})$ to obtain the wavelet estimate $\hat{f}(x_i)$.

Let $J_n = log_2\,(n)/\{5 - 2\,\Delta(f) + \eta\}$, with η any positive constant, and select integers j_n such that $j_n \leq J_n$ and $j_n \to \infty$ as $n \to \infty$. The following theorem shows that for large n the fractal dimension of $f(x)$ is almost preserved in the thresholded empirical wavelet coefficients $\delta_{t_n}(y_{j,k})$ at resolution levels j_n.

Theorem 1 *Under (8) we have that as $n \to \infty$,*

$$\frac{3}{2} + log_2 \left(\sum_k |\delta_{t_n}(y_{j_n,k})| \right) \Big/ j_n \to \Delta(f) \quad \textit{in probability.} \tag{10}$$

Proof. Following Donoho (1993a, 1994, 1995, section 5) we have that

$$y_{j,k} = \theta_{j,k} + n^{-1/2}\, \varepsilon_{j,k}, \qquad j = 0, \cdots, J, \;\; k = 0, \cdots, 2^j - 1, \tag{11}$$

where $(y_{j,k})$ and $(\varepsilon_{j,k})$ are the empirical wavelet coefficients of (y_i) and (ε_i), respectively. The thresholded wavelet coefficients $\delta_{t_n}(y_{j,k})$ satisfy

$$\sum_k |y_{j,k}| \geq \sum_k |\delta_{t_n}(y_{j,k})| \geq \sum_k |y_{j,k}| - 2^j\, t_n. \tag{12}$$

The random errors ε_i are independent normal random variables with mean zero and variance σ^2, and the wavelet transformation is orthogonal, so the $\varepsilon_{j,k}$ are also independent normal random variables with mean zero and variance σ^2. A direct application of the Chebyshev inequality shows that $\sum_{k=1}^{2^{j_n}} |\varepsilon_{j_n,k}|$ is of order 2^{j_n}. By (11) we have that as $n \to \infty$,

$$
\begin{aligned}
&log_2 \left(\sum_k |y_{j_n,k}| \right) = log_2 \left(\sum_k \left| \theta_{j_n,k} + n^{-1/2}\, \varepsilon_{j_n,k} \right| \right) \\
&= log_2 \left\{ \sum_k |\theta_{j_n,k}| + O_p \left(n^{-1/2}\, 2^{j_n} \right) \right\} \\
&= log_2 \left(\sum_k |\theta_{j_n,k}| \right) + log_2 \left\{ 1 + O_p \left(n^{-1/2}\, 2^{j_n} \Big/ \sum_k |\theta_{j_n,k}| \right) \right\} \\
&= log_2 \left(\sum_k |\theta_{j_n,k}| \right) + O_p \left(n^{-1/2}\, 2^{j_n} \Big/ \sum_k |\theta_{j_n,k}| \right), \qquad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
&log_2 \left(\sum_k |y_{j_n,k}| - 2^{j_n}\, t_n \right) = log_2 \left\{ \sum_k |\theta_{j_n,k}| + O_p \left(n^{-1/2}\, 2^{j_n} \right) - 2^{j_n}\, t_n \right\} \\
&= log_2 \left(\sum_k |\theta_{j_n,k}| \right) + log_2 \left\{ 1 + O_p \left((log\, n)^{1/2}\, n^{-1/2}\, 2^{j_n} \Big/ \sum_k |\theta_{j_n,k}| \right) \right\} \\
&= log_2 \left(\sum_k |\theta_{j_n,k}| \right) + O_p \left((log\, n)^{1/2}\, n^{-1/2}\, 2^{j_n} \Big/ \sum_k |\theta_{j_n,k}| \right). \qquad (14)
\end{aligned}
$$

Equation (8) and the selection of j_n imply that

$$
\begin{aligned}
&\lim_{n \to \infty} \frac{(log\, n)^{1/2}\;\; n^{-1/2}\, 2^{j_n} / \sum_k |\theta_{j_n,k}|}{j_n} \qquad (15) \\
&\quad = \lim_{n \to \infty} \left\{ \left(log\, n / j_n^2 \right)^{1/2} \;\; 2^{(\{5 - 2\Delta(f) + \gamma(n)\}\, j_n - log_2 n)/2} \right\} \to 0,
\end{aligned}
$$

where $\gamma(n) \to 0$ as $n \to \infty$. Substituting (15) into (13) and (14) and using (8), we obtain

$$
log_2 \left(\sum_k |y_{j_n,k}| \right) \Big/ j_n \to \Delta(f) - 3/2 \quad \text{in probability,}
$$

and

$$
log_2 \left(\sum_k |y_{j_n,k}| - 2^{j_n}\, t_n \right) \Big/ j_n \to \Delta(f) - 3/2 \quad \text{in probability.}
$$

The theorem immediately follows from (12) and above two results. □

The wavelet coefficients $\theta_{j,k}$ at high resolution levels describe the detailed features of $f(x)$ and determine its fractal dimension. However, random errors often distort or even destroy $\theta_{j,k}$ at very high resolution levels. For $f(x)$ satisfying the Hölderian condition with exponent α, (7) implies that $\theta_{j,k}$ is of order $2^{-j\,(\alpha+1/2)}$. As noise term $n^{-1/2}\,\varepsilon_{j,k}$ in (11) is of order $n^{-1/2}$, from (11) we easily conclude that at very high resolution levels, $y_{j,k}$ is dominated by the noise term, and thus the very detailed feature of $f(x)$ can't be well recovered. On the other hand, at relatively lower resolution levels j_n, $\theta_{j_n,k}$ controls $y_{j_n,k}$, and hence, as proved in Theorem 1, for large n the fractal dimension of $f(x)$ is nearly preserved in the thresholded wavelet coefficients $\delta_{t_n}(y_{j_n,k})$ at resolution levels j_n. In view of (8), we interpret (10) as implying that as n tends to infinity, the fractal dimension of wavelet estimate $\hat{f}$ at resolution levels j_n converges in probability to $\Delta(f)$. Applying the algorithm described near the end of section 2 to $\hat{f}(x_i)$ and choosing ϵ to range over scales at these resolution levels, we yield a fractal dimension estimate close to $\Delta(f)$. Therefore, for large samples, the fractal dimension of $f(x)$ is well preserved in the wavelet estimate. In contrast, estimates constructed by smoothing methods always have dimension one, and it is impossible to recover the fractal dimension of the underlying function from the smoothing estimates.

Condition (8) may be very difficult to verify in practice. However, as in fractal dimension calculations, its sufficient conditions (3) and (4) are often fairly easy to check.

Restricting $f(x)$ and data to a small neighborhood of x_0, we can establish a localized version of (10) and show that, for large samples, the wavelet estimate $\hat{f}$ at some resolution levels has local dimension at x_0 close to $\Delta(f, x_0)$.

5 Examples

Two simulated examples were carried out to check the performance of the method. In the first example we took the underlying function to be the Weierstrass function with fractal dimension 1.5,

$$f(x) = \sum_{k=0}^{\infty} 1.5^{-0.5\,k}\, sin(1.5^k \cdot 5\,x), \qquad x \in [0, 1]. \tag{16}$$

The Weierstrass function is self-affine and has equal local dimensions (Massopust 1994; Tricot 1995).

The function $f(x)$ was evenly sampled $n = 1024$ times and i.i.d. normal random errors were then added. Since Gaussian white noise has fractal dimension 2, the noisy function (the function plus noise) has fractal dimension 2 (Tricot, 1995, p. 149). Daubechies wavelet with two vanishing moments was used in construction of the wavelet estimate. The true function, noisy function, and wavelet estimate are plotted in Figure 1. The visual impression of the figure indicates that wavelet shrinkage removed most of noise from the noisy data, and the wavelet estimate appears to be as rough as the true function. Using the algorithm described at the end of section 2, we computed the fractal dimensions of the true, estimated and noisy curves to be 1.49, 1.47 and 1.96, respectively. In the computation m was taken to be 85 and ϵ_i were equally spaced in an interval from 0.07 to 0.45. The numerical results evidently confirm our intuitive conclusion.

In the second example, $f(x)$ was chosen to be a sample path of the Gaussian process

$$X(x) = X_0 + \int_0^x (x-t)^{H(x)-1/2}\, dB_t, \qquad x \in [0,1], \tag{17}$$

where X_0 is a standard normal random variable, B_t is a standard Brownian motion independent of X_0, and

$$H(x) = 0.3 + (0.2 - 0.4\,x)^{0.5}\, 1(0 \le x \le 0.5) + (0.4\,x - 0.2)^{0.6}\, 1(0.5 < x \le 1).$$

$X(x)$ generalizes fractional Brownian motion by letting the self-similarity parameter H vary with x. The plot of the sample path in Figure 2 (a) clearly indicates that it has varying local dimensions and is no longer statistically self-similar.

As in Example 1, $f(x)$ was sampled 1024 times, the noisy data were obtained by adding i.i.d. normal errors, and the wavelet estimate was then computed. The true, noisy and estimated curves were plotted in Figure 2, and their fractal dimensions were computed to be 1.50, 1.90 and 1.42, respectively. Wavelet shrinkage successfully removed noise from $f(x)$ without significantly changing its fractal dimension; moreover, the plot of the wavelet estimate in Figure 2(c) obviously reveals the varying local dimensions of $f(x)$.

6 Discussion

Limits like (10) usually have slow convergence rates. The proof of Theorem 1 indicates that the convergence rate in (10) heavily depends on the speed of convergence in (8).

Although the limits in both (8) and (10) are independent of equal local dimensions, self-affinity and self-similarity, such homogeneous properties often speed up the convergence of (8) and therefore (10). For functions with such properties, asymptotic behaviors may be quickly achieved and our method tends to yield better numerical results. This is apparently supported by the simulations. Despite slightly lower signal to noise ratio, Example 1 estimated fractal dimension more accurately than Example 2. The phenomenon seems largely due to the fact that the Weierstrass function has equal local dimensions and self-affinity while the function in the second example has neither property. Of course, to clarify all these issues, we need to derive the convergence rate for the limit in (10), and in particular study how these properties affect the convergence rate and how much impact they have on finite sample properties.

Also slow convergence suggests that (10) is generally not good for directly estimating fractal dimension. The remarks of a referee lead us to think of combining (10) with a regression method to calculate fractal dimension of $\hat{f}$ as follows. Select some resolution levels j, compute $log_2(\sum_k |\delta_{t_n}(y_{j,k})|)$ at these levels and then regress the computed values on j. The fractal dimension can then be estimated by the regression slope plus 3/2. However, it is not clear how efficient this algorithm will be. Presumably the performance ultimately hinges on the speed of convergence in (10). This is another area of future research.

Acknowledgements

The research is supported by NSF Grant and Research Council Grant at the University of Missouri-Columbia. The author would like to thank Paul Speckman, an anonymous referee, an Associate Editor and the Editor for their comments that led to significant improvement of the paper.

REFERENCES

Adler, R. L. (1981) The Geometry of Random Fields. New York: Wiley.

Barnsley, M. F. (1993) Fractals Everywhere. New York: Academic Press.

Chorafas, D. N. (1994) Chaos Theory in the Financial Markets: Applying Fractals, Fuzzy Logic, Genetic Algorithms, Swarm Simulation & the Monte Carlo Method to Manage Market Chaos & Volatility. Chicago: Probus.

Cohen, A., Daubechies, I., Jawerth, B. and Vial, P. (1993) Multiresolution analysis, wavelets, and fast algorithms on an interval. Comptes Rendus Acad. Sci. Paris (A), **316**, 417-421.

Constantine, A. G. and Hall, P. (1994) Characterizing surface smoothness via estimation of effective fractal dimension. J. Roy. Statist. Soc. B, **56**, 97-113.

Daubechies, I. (1992) Ten Lecture on Wavelets. Philadelphia: SIAM.

Daubechies, I. (1994) Two recent results on wavelets: Wavelet bases for the interval, and biorthogonal wavelet diagonalizing the derivative operator. In Recent Advances in Wavelet Analysis (eds. L. L. Schumaker and G. Webb), pp. 237-257. New York: Academic Press.

Donoho, D. L. (1993a) Smooth wavelet decompositions with blocky coefficient kernels. In Recent Advances in Wavelet Analysis (eds. L. L. Schumaker and G. Webb), pp.

259-308. New York: Academic Press.

Donoho, D. L. (1993b) Unconditional bases are optimal bases for data compression and for statistical estimation. Appl. Comput. Harmon. Anal., **1**, 100-115.

Donoho, D. L. (1994) Interpolating wavelet transformations. Technical report, Dept. of Statist., Stanford Univ.

Donoho, D. L. (1995) De-noising by soft thresholding. IEEE Trans. Inform. Theory, **41**, 613-627.

Donoho, D. L. and Johnstone, I. M. (1994) Ideal spatial adaptation by wavelet shrinkage. Biometrika, **81**, 425-455.

Donoho, D. L. and Johnstone, I. M. (1995) Adapting to unknown smoothness via wavelet shrinkage. J. Amer. Statist. Assoc., **90** 1200-1224.

Donoho, D. L. and Johnstone, I. M. (1996) Minimax estimation via wavelet shrinkage. To appear in Ann. Statist.

Donoho, D. L., Johnstone, I. M., Kerkyacharian, G. and Picard, D. (1995) Wavelet shrinkage: Asymptopia ? (with discussion). J. Roy. Statist. Soc. B, **57**, 301-369.

Farge, M., Hunt, C. R. and Vassilicos, J. C. (1993) Wavelets, Fractals and Fourier Transformations. Oxford: Clarendon Press.

Grassberger, P., Hegger, R., Kantz, H., Schaffrath, C. and Schreiber, T. (1993) On noise reduction methods for chaotic data. Chaos, **3**, 127-142.

Hall, P. (1995) On the effect of measuring a self-similar process. SIAM J. Appl. Math., **55**, 800-808.

Hall, P. and Wood, A. (1993) On the performance of box-counting estimators of fractal dimension. Biometrika, **80**, 246-252.

Jaffard, S. (1994) Wavelets and nonlinear analysis. In Wavelets: Mathematics and Applications (eds. J. J. Benedetto and M. W. Frazier), pp. 467-503. Ann Arbor: CRC Press.

Kostelich, E. and J. Yorke, J. A. (1988) Noise reduction in dynamical systems. Phys. Rev. A, **38**, 1469-1473.

Mandelbrot, B. B. (1983) The Fractal Geometry of Nature. Updated and augmented edition. New York: Freeman.

Massopust, P. R. (1994) Fractal Functions, Fractal Surfaces, and Wavelets. New York: Academic Press.

Meyer, Y. (1992) Wavelets and Operators. Oxford: University Press.

Ogata, Y. and Katsura, K. (1991) Maximum likelihood estimates of the fractal dimension for random spatial patterns. Biometrika, **78**, 463-474.

Peitgen, H. O., Jurgens, H. and Saupe, D. (1992) Chaos and Fractals: New Frontiers of Science. New York: Springer-Verlag.

Taylor, C. C. and Taylor, S. J. (1991) Estimating the dimension of a fractal. J. Roy. Statist. Soc. B, **53**, 353-364.

Tricot, C. (1995) Curves and Fractal Dimension. New York: Springer-Verlag.

Wang, Y. (1995) Jump and sharp cusp detection by wavelets. Biometrika, **82**, 385-397.

Wang, Y. (1996) Function estimation via wavelet shrinkage for long-memory data. Ann.

Statist. **24**, 466-484.

Wornell, G. W. (1990) A Karhunen-Loéve-like expansion for $1/f$ processes via wavelets. IEEE Trans. Inform. Theory, **36**, 859-861.

Wornell, G. W and Oppenheim, A. V. (1992) Estimation of fractal signals from noisy measurements using wavelets. IEEE Trans. Signal Process, **40**, 611-623.

Department of Statistics

University of Missouri-Columbia

Columbia, Missouri 65211

U. S. A.

Legend for Figures

Figure 1. The data are simulated from model $y_i = f(i/n) + \varepsilon_i$, $i = 1, \cdots, n = 1024$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0.2$, $f(x)$ is the Weierstrass function given by (16); (a), (b) and (c) are the true function, the noisy function and the wavelet estimate, respectively.

Figure 2. The data are simulated from model $y_i = f(i/n) + \varepsilon_i$, $i = 1, \cdots, n = 1024$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0.2$, $f(x)$ is a sample path of Gaussian process defined by (17); (a), (b) and (c) are the true function, the noisy function and the wavelet estimate, respectively.