Small Ball Problem via Wavelets for Gaussian Processes

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Abstract

The recently developed theory of wavelets is used to estimate small ball probability for a class of Gaussian processes including d-dimensional fractional Brownian motion.

Key words: Brownian motion, fractional Brownian motion, Gaussian process, small ball problem, wavelets, wavelet-vaguelette decomposition.

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1 Introduction

The recently developed theory of wavelets is a breakthrough in applied mathematics with potential applications in a wide range of disciplines. Benassi and Jaffard (1993) used wavelets to decompose Gaussian processes. A class of Gaussian processes including Brownian motion and fractional Brownian motion are associated with dilation-homogeneous operators, which can be diagonalized through wavelet-vaguelette decompositions [Meyer 1990, Frazier, Jawerth, and Weiss 1991, Donoho 1995] and the corresponding processes can then be decomposed into a series of i.i.d. normal random variables [see Benassi and Jaffard 1993]. By use of wavelet-decompositions Benassi and Jaffard (1993) studied regularity of Gaussian processes. In this paper we use wavelet methods to estimate small ball probability for a wide class of Gaussian processes including d dimensional Brownian motion, d dimensional fractional Brownian motion and processes with nonstationary increments.

Estimates on small ball probability are very useful in theory of probability and statistics. Bass (1988) used Haar basis decompose Brownian motion and then derived estimates on small ball probability for Brownian motion. Talagrand (1994) provided a sharp bound on small ball probability for Brownian sheet. Monrad and Rootzén (1995) studied small ball problem for locally nondeterministic Gaussian processes including one dimensional fractional Brownian motion. Bass (1988) and Talagrand (1994) both employed Haar basis to decompose Brownian motion and their methods have a wavelet flavor. However, Haar basis can't be used to deal with processes other than Brownian motion. This paper shows that the recently developed theory of wavelets can handle a class of Gaussian processes including d dimensional fractional Brownian motion.

The paper is organized as follows. Section 2 presents main results and Section 3 provides proofs. Remarks are put in Section 4.

2 Small Ball Probability

Suppose X is a Gaussian process with a reproducing kernel space H defined by a symmetric positive operator A as follows:

$$H = \{f \in L^2(\mathbb{R}^d), \ \ [f,f]_H < \infty\}, \tag{1}$$

where

$$[f,g]_H = \int (Af)(x)\, g(x) dx, \qquad Af(x) = \int e^{i\,x\,\omega}\, \sigma(x,\omega)\, \hat{f}(\omega)\, d\omega, \tag{2}$$

$\sigma : I\!R^d \times I\!R^d \to I\!R$ satisfies that for $m > 0$,

$$C_1 |\omega|^m + C_2 \leq \sigma(x,\omega) \leq C_3 |\omega|^m + C_4, \tag{3}$$

for some positive constants C_1, C_2, C_3 and C_4, and the smoothness condition: with $s = (s_1, \cdots, s_d) \in I\!N^d$, $|s| = s_1 + \cdots s_d$ and

$$\partial^s = \frac{\partial^{s_1}}{\partial x_1^{s_1}} \cdots \frac{\partial^{s_d}}{\partial x_d^{s_d}},$$

for $r \in I\!R$ and $n \in I\!N$,

$$|\partial_\omega^s \partial_x^{s'} \sigma(x,\omega)| \leq C\{1 + |\omega|\}^{m+|s|-|s'|}, \qquad |s| \leq n \text{ and } |s'| \leq r,$$

$$|\partial_\omega^s \partial_x^{s'} \{\sigma(x,\omega) - \sigma(x_0,\omega)\}| \leq C\{1 + |\omega|\}^{m+r-|s|} |x - x_0|^{r-[r]}, \qquad |s| \leq n \text{ and } |s'| = [r],$$

$$|\sigma(x,\omega) - \sigma(x_0,\omega)| \leq C(1 + |\omega|)^{m+\zeta'} |x - x_0|^{\zeta}, \qquad \zeta > \zeta' \geq 0.$$

The above definition covers a large class of Gaussian process including Brownian motion and fractional Brownian motion. For example, for ordinary d-dimensional Brownian motion, $A = (-\Delta)^{1/2+d/2}$ and $m = d+1$, where Δ is the Laplacian, and for fractional Brownian motion with index α, $A = (-\Delta)^{\alpha+d/2}$ and $m = d + 2\alpha$. The following theorem provide estimates on small ball probability for the process X.

Theorem 1 *There exist positive constants c and C such that for small $\delta > 0$,*

$$exp\left\{-\frac{c}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^{3(d-1)}\right\} \leq P\left(\sup_{x\in[0,1]^d} |X(x)| \leq \delta\right) \leq exp\left\{-\frac{C}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^{d-1}\right\},$$

with $\alpha = (d-m)/2$.

The left inequality in (3), the uniformly elliptic condition, is required by the upper bound in Theorem 1 while the right inequality is needed to establish the lower bound.

Define the norm

$$\|X\|_2 = \left(\int_{[0,1]^d} |X(x)|^2 dx\right)^{1/2}.$$

Theorem 2 *If $\sigma(x,\omega) = b_1 \omega^{b_2}$ for some positive constants b_1 and b_2, then there exists a positive constant C such that for small $\delta > 0$,*

$$P(\|X\|_2 \leq \delta) \leq exp\left\{-\frac{C}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^{2(d-1)}\right\},$$

with $\alpha = (d-m)/2$.

Applying Theorem 1 and Theorem 2 to fractional Brownian motion, we obtain the following corollary.

Corollary 1 *For d-dimensional fractional Brownian motion B_α,*

$$exp\left\{-\frac{c}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^{3(d-1)}\right\} \le P\left(\sup_{x\in[0,1]^d}|B(x)| \le \delta\right) \le exp\left\{-\frac{C}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^{2(d-1)}\right\}.$$

To the best of our knowledge, even the result for d dimensional fractional Brownian motion is new.

Taking $d = 1$ in Corollary 1 we have

$$exp\left(-\frac{c}{\delta^{1/\alpha}}\right) \le P\left(\sup_{x\in[0,1]}|B_\alpha(x)| \le \delta\right) \le exp\left(-\frac{C}{\delta^{1/\alpha}}\right),$$

which are the exact bounds obtained in Monrad and Rootzén (1995) for 1-dimensional fractional Brownian motion.

We now consider an one dimensional process $X(x)$, $x \in I\!R$ without satisfying elliptic condition: Take

$$\sigma(x,\omega) = a(x)\,\omega^2 - a'(x)\,\omega, \qquad \text{then } A = -\frac{d}{dx}\left\{a(x)\,\frac{d}{dx}\right\}. \tag{4}$$

where $a(x)$ is a positive continuous function with $a'(x)$ the generalized derivative of $a(x)$. If $a(x)$ is bounded below from zero and above from infinity, then (3) is satisfied. However, if $a(x) \sim |x|^\beta$ as $x \to 0$ with $0 < \beta < 1$, and $a(x)$ is continuous and $a(x) > 0$ except for $x = 0$, then as x near 0, the elliptic condition [the left inequality in (3)] is violated.

Theorem 3 *For the process defined by (4), there exist positive constants c and C such that*

$$exp\left(-\frac{c}{\delta^{1/\alpha}}\right) \le P\left(\sup_{x\in I}|X(x)| \le \delta\right) \le exp\left(-\frac{C}{\delta^{1/\alpha}}\right),$$

with $\alpha = (1-\beta)/2$ and I a neighborhood of $x = 0$.

3 Wavelet Decomposition

Benassi and Jaffard (1993) decomposed the Gaussian process X defined in Section 1 into a series of i.i.d. normal random variables through wavelets. Let $E = I\!R^d$, J_d be the collection of all cubes λ centered at $k\,2^{-j}$ with side length 2^{-j}, $j \in Z\!\!Z$, $k \in Z\!\!Z^d$. Put $|\lambda| = j$. Denote by τ_d the 2^dadic tree with knots coded by J_d and $\{\varepsilon_\lambda, \lambda \in J_d\}$ the white noise on the tree τ_d [that is, we set i.i.d. normal variable at each knot of the tree]. For $d = 1$, $E = I\!R$, $\lambda = (j,k)$, $j \in I\!N$, $k \in I\!N$, $1 \leq k \leq 2^j$, J_1 is the collection of all these $\lambda = (j,k)$, and τ_1 is the dyadic tree with knots coded by J_1.

Let ψ_λ be a (tensor product) wavelet orthogonal basis of $L^2(I\!R^d)$, with compact support and sufficient smoothness [see Daubechies 1992, Meyer 1990]. Let

$$u_\lambda(x) = A^{-1/2}(\psi_\lambda)(x). \tag{5}$$

Then u_λ will form a wavelet orthogonal basis of the reproducing kernel space H, with compact support, and obey

$$C_1\, 2^{j(m-d)/2} \leq \sup_{x \in E} |u_\lambda(x)| \leq C_2\, 2^{j(m-d)/2}, \tag{6}$$

and X can be decomposed as

$$W(x) = \sum_{\lambda \in J_d} \varepsilon_\lambda\, u_\lambda(x). \tag{7}$$

For the process defined by (4) with $a(x) \sim |x|^\beta$ as $x \to 0$. The maximum of u_λ over a neighborhood I of $x = 0$ satisfies

$$C_1\, 2^{j(\beta-1)/2} \leq \sup_{x \in I} |u_\lambda(x)| \leq C_2\, 2^{j(\beta-1)/2}. \tag{8}$$

See Benassi and Jaffard (1993).

Theorem 4 *Assume $\sigma(x,\omega) = b_1\, \omega^{b_2}$ for some positive constants b_1 and b_2. Then u_λ [after normalized by constant of order $2^{j\,\alpha}$] is a nearly orthogonal vaguelette system, that is,*

$$C_1 \sum_\lambda 2^{-2\,n\,\alpha}\, |a_\lambda|^2 \leq \int_{E^d} |\sum_\lambda a_\lambda\, u_\lambda(x)|^2\, dx \leq C_2 \sum_\lambda 2^{-2\,n\,\alpha}\, |a_\lambda|^2. \tag{9}$$

In fact, if define

$$v_\lambda(x) = A^{1/2}\psi_\lambda(x),$$

then $(\psi_\lambda)_\lambda$, $(u_\lambda)_\lambda$ and $(v_\lambda)_\lambda$ [after normalized] form a wavelet-vaguelette decomposition of $A^{1/2}$, that is,

(1) Three sets of functions $(\psi_\lambda)_\lambda$ – an orthogonal wavelet basis – and $(u_\lambda)_\lambda$ and $(v_\lambda)_\lambda$ – near-orthogonal vaguelettes.

(2). Quasi-singular value relations $A^{1/2}\psi_\lambda = \kappa_j v_\lambda$ and $(A^{1/2})^* u_\lambda = \kappa_j \psi_\lambda$ for $\lambda = (j,k)$ with quasi-singular values (κ_j), depending on resolution index j but not spatial index k.

(3). Biorthogonality relations $[u_\lambda, v_\mu] = \delta_{\lambda,\mu}$.

[For details see Donoho (1995).]

Proof. We will prove only the near-orthogonal (9) of u_λ, since the rest of proofs can be easily checked [e.g. see Wang (1994)]. The Fourier transformations of $A^{1/2}\psi$ and $A^{-1/2}\psi$ are $b_1\,\hat{\psi}(\omega)\,|\omega|^{b_2+1/2}$ and $b_1\,\hat{\psi}(\omega)\,|\omega|^{-b_2-1/2}$, respectively, and hence they are vaguelettes by Definition 1 and Lemma 4 of Donoho (1995). Therefore, the near-orthogonality of $A^{1/2}\,\psi_\lambda$ and $A^{-1/2}\,\psi_\lambda$ is a consequence of Theorem 2 of Donoho (1995) [see also Meyer (1990) and Wang (1994)]. $\Box$

4 Proofs

The proof of Theorem 3 is similar to that of Theorem 1 except for using (8) instead of (6), so we omit its proof. In the following proofs C stands for generic constants whose values many change from line to line.

4.1 Upper Bound of Theorem 1

The proof is similar to that of Proposition 3.3 of Bass [1988, p. 59], we sketch it only. By (6) and (7) we have that $\sup_{x\in E^d}|X(x)| \le \delta$ implies $|\xi_\lambda| = |\int v_\lambda(x)X(x)dx| \le C\sup_{x\in E^d}|v_\lambda(x)\,X(x)| \le C\,2^{j(d-m)/2}\,\delta = C\,2^{j\,\alpha}\,\delta$. Therefore,

$$P\left(\sup_{x\in E^d}|X(x)| \le \delta\right) \le \prod_{j\le M}\prod_{|\lambda|=j} P\left(|\xi_\lambda| \le C\,2^{j\,\alpha}\,\delta\right).$$

For standard normal random variable ξ_λ,

$$P\left(|\xi_\lambda| \le C\,2^{j\,\alpha}\,\delta\right) \le C\,2^{j\,\alpha}\,\delta.$$

Then

$$P\left(\sup_{x\in E^d}|X(x)| \le \delta\right) \le \prod_{j\le M}\prod_{|\lambda|=j} C\,2^{j\,\alpha}\,\delta$$
$$\le \prod_{j\le M}\left(C\,2^{j\,\alpha}\,\delta\right)^{j^{d-1}\,2^j}.$$

Choosing M such that $C\,2^{M\alpha}\,\delta \le 1$ [i.e. $M =$ the integer part of $-log_2(C\,\delta)/\alpha$] and taking logarithms, we obtain

$$\begin{aligned}
&log\left\{P\left(\sup_{x\in E^d}|X(x)|\le\delta\right)\right\}\le\sum_{j=0}^{M} j^{d-1}\;2^j\;log\left(C\,2^{j\,\alpha}\,\delta\right)\\
&\le C_1\int_0^M t^{d-1}\,2^t\,log\left(C\,2^{t\alpha}\,\delta\right)dt\\
&\le C_1\int_\delta^{2^M}(log x)^{d-1}\,log(C\,x^\alpha\,\delta)dx\\
&\le C_1\int_\delta^{2^M}(log x)^{d-1}\,log(C_2\,x\,\delta^{1/\alpha})dx\\
&\le C_1\,\delta^{-1/\alpha}\int_{\delta^{1/\alpha}}^{2^M\,\delta^{1/\alpha}}\left\{(-log\,\delta)^{d-1}+(log\,u)^{d-1}\right\}\,log(C_2\,u)du\\
&\le -C\,\delta^{-1/\alpha}\,\{log(1/\delta)\}^{d-1}\,.
\end{aligned}$$

4.2 Lower Bound of Theorem 1

Fix δ small, and choose N so that $2^N\,\delta^2/N^{2(d-1)}\in[1,16]$. Select

$$\beta_\lambda=\{C\,(|\lambda|+4)\}\,(|N-|\lambda|\,|+1)^{-2}$$

such that

$$\sum_\lambda\beta_\lambda=\sum_{j=0}^{\infty}\sum_{|\lambda|=j}\beta_\lambda\le 1.$$

u_λ are compactly supported, so each x will be in the supports of a fixed number of those $u_\lambda(x)$ with $|\lambda|=j$. Hence,

$$\sup_{x\in E^d}|\sum_{|\lambda|=j}u_\lambda(x)\,\xi_\lambda|\le C\sup_{|\lambda|=j}\;\sup_{x\in E^d}|u_\lambda(x)|\;|\xi_\lambda|. \tag{10}$$

If $\sup_{|\lambda|=j}\sup_{x\in E^d}|u_\lambda|\;|\xi_\lambda|<\beta_\lambda\,\delta$ for each λ, then by (7) and (10) we get

$$\sup_{x\in E^d}|X(x)|\le\sum_\lambda\sup_{x\in E^d}|\sum_{|\lambda|=j}u_\lambda(x)\,\xi_\lambda|\le\sum_\lambda\beta_\lambda\delta\le\delta,$$

and hence

$$\begin{aligned}
&P(\|X(x)\|\le\delta)\ge\prod_\lambda P\left(\sup_{|\lambda|=j}\sup_{x\in E^d}|u_\lambda(x)|\,|\xi_\lambda|\le\beta_\lambda\,\delta\right)\\
&=\prod_\lambda P\left(\sup_{|\lambda|=j}|\xi_\lambda|\le C\,2^{j\,\alpha}\,\beta_\lambda\,\delta\right)\qquad\text{by (6)}\\
&=\prod_j\prod_{|\lambda|=j}P\left(\sup_{|\lambda|=j}|\xi_\lambda|\le C\,2^{j\,\alpha}\,\beta_\lambda\,\delta\right).
\end{aligned}$$

The rest of proof is similar to that of Proposition 4.4 of Bass (1988).

4.3 Proof of Theorem 2

By (7) and Lemma 2.1 of Talagrand [1994, p. 1335] we obtain

$$P\left(\int_{E^d} |X(x)|^2 dx \le \delta^2\right) \;\le\; P\left(\int_{E^d} |\sum_{|\lambda|=n} u_\lambda(x)\,\xi_\lambda|^2\,dx \le \delta^2\right).$$

Near-orthogonality of $2^{j\,\alpha}\,u_\lambda(x)$ proved in Lemma 1 implies

$$C_1 \sum_{|\lambda|=n} 2^{-2\,n\,\alpha}\,|\xi_\lambda|^2 \le \int_{E^d} |\sum_{|\lambda|=n} u_\lambda(x)\,\xi_\lambda|^2\,dx \le C_2 \sum_{|\lambda|=n} 2^{-2\,n\,\alpha}\,|\xi_\lambda|^2.$$

Then

$$\begin{aligned} P\left(\int_{E^d} |X(x)|^2 dx \le \delta^2\right) &\le P\left(\sum_{|\lambda|=n} 2^{-2\,n\,\alpha}\,|\xi_\lambda|^2 \le C\,\delta^2\right) \\ &\le P\left(\sum_{|\lambda|=n} |\xi_\lambda|^2 \le C\,\delta^2\,2^{2\,n\,\alpha}\right). \end{aligned}$$

Note that $card\{\lambda, |\lambda| = n\} \sim n^{k-1}\,2^n$. Taking $n^{k-1} 2^{-n} \sim \delta^{1/\alpha}$ and by Chebyshev's inequality we obtain

$$P\left(\sum_{|\lambda|=n} |\xi_\lambda|^2 \le C\,\delta^2\,2^{2\,n\,\alpha}\right) \le exp\left\{-\frac{C}{\delta^{1/\alpha}}\left(\log\frac{1}{\delta}\right)^{2(k-1)}\right\}.$$

5 Concluding Remark

Talagrand (1994) established a sharp upper bound on small ball probability for two-dimensional Brownian motion: there exist a positive constant C such that for small $\delta > 0$,

$$exp\left\{-\frac{C}{\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^3\right\} \le P\left(\sup_{x\in[0,1]^2} |X(x)| \le \delta\right) \le exp\left\{-\frac{1}{C\,\delta^{1/\alpha}}\left(log\frac{1}{\delta}\right)^3\right\}.$$

For two-dimensional Brownian motion, $A = \Delta$. If ψ_λ is Haar basis, then u_λ are integrals of Haar basis. The derivation of the sharp upper bound relies on an inequality for integrals of Haar basis [see Theorem 3.1 of Talagrand (1994)]. However, we fail to establish a similar inequality for general wavelets and hence a sharp upper bound for general Gaussian processes in several dimensions. Such inequality holds for Haar basis and integrals of Haar basis and we conjecture it also holds for some smooth wavelet bases. Hopefully future wavelet research will shed light on it.

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