(Near-)optimal Results for Phase Synchronization

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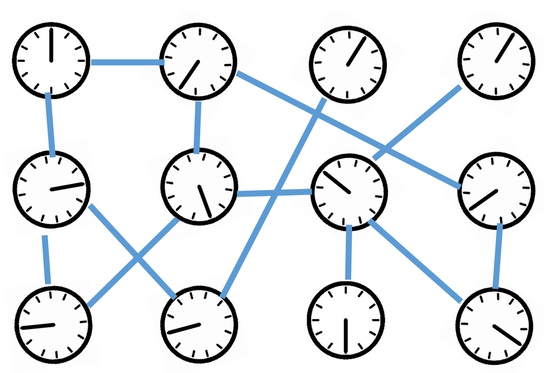
Phase (angular) synchronization

Problem formulation: estimate unknown parameters (angles) $\theta_1, \theta_2, \ldots, \theta_n \in [0, 2\pi)$ based on pairwise measurements:

$$y_{\ell k} = \text{noisy version of } \theta_\ell - \theta_k \mod 2\pi,$$

where $1 \le \ell < k \le n$.

Example: Time synchronization



Model: modeling signal $z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ and data

$$C_{\ell k} = \bar{z}_\ell z_k + \sigma W_{\ell k}, \qquad \forall \ell > k$$

where $W_{\ell k} \sim N_{\mathbb{C}}(0, 1)$; or in matrix form

$$C = zz^* + \sigma W, \quad \text{with } |z_k| = 1, \ \forall\, k \in [n].$$

Nonconvexity: nonconvex constraints; hard to study MLE.

Taming nonconvexity?

Consider the standard recipe—semidefinite program (SDP) relaxation: the MLE $\hat{x}$ is a solution to

$$\max_{X \in \mathbb{C}^{n\times n}, X = X^*} \operatorname{Tr}(CX)$$
$$\text{subject to } \operatorname{diag}(X) = \mathbf{1}, X \succeq 0. \ \sout{\operatorname{rank}(X) = 1}$$

Observations: low rank (signal matrix) + random noise

- the 'signal' zz^* is a rank-one matrix with $\lambda_{\max}(zz^*) = n$,
- the 'noise' has magnitude $\|W\| = \Theta(\sqrt{n})$ w.h.p.,
- we expect phase transition occurs at $\sigma = \Theta(\sqrt{n})$ (see below).

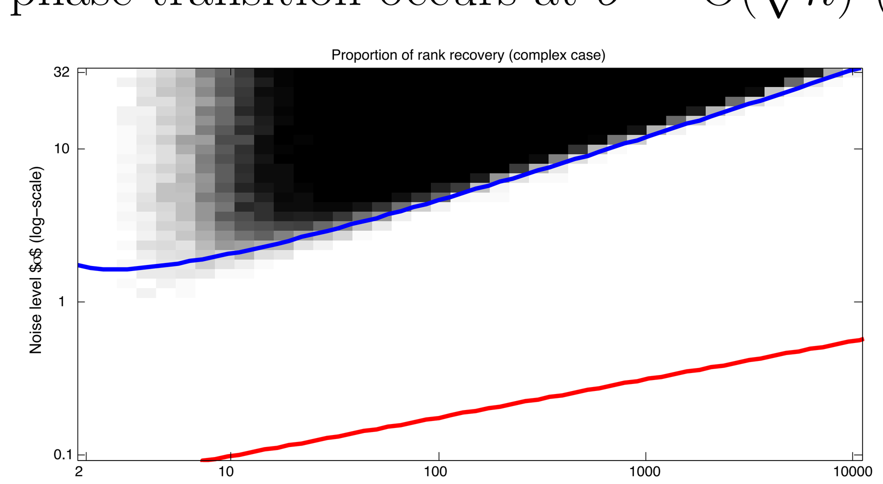


Figure 1: Red: $\sigma = \frac{1}{18} n^{1/4}$; Blue: $\sigma = \sqrt{\frac{n}{\log n}}$ (Figure from [2])

Difficulty: however, previous works can only do $\sigma = O(\sigma^{1/4})$.

- Hard to analyze statistical dependence between $\hat{x}$ and W.
- More generally, how to study randomness with nonconvexity?

Convergence analysis via decoupling sequences

Generalized power method (GPM): A simple and fast approach.

(1) Set x^0 to be a leading eigenvector of C with $\|x^0\|_2 = \sqrt{n}$.

(2) For $t = 0, 1, \ldots$, update $(x^{t+1})_k = \dfrac{(Cx^t)_k}{|(Cx^t)_k|}$.

▲ Decoupling analysis of GPM gives new algorithmic/statistical understanding.

Algorithmic guarantee:

> If $\sigma = \mathcal{O}\left(\sqrt{n/\log n}\right)$, with high probability for large n, SDP admits a unique solution $\hat{x}\hat{x}^*$, and GPM converges linearly to $\hat{x}$ (up to phase).

Statistical guarantee:

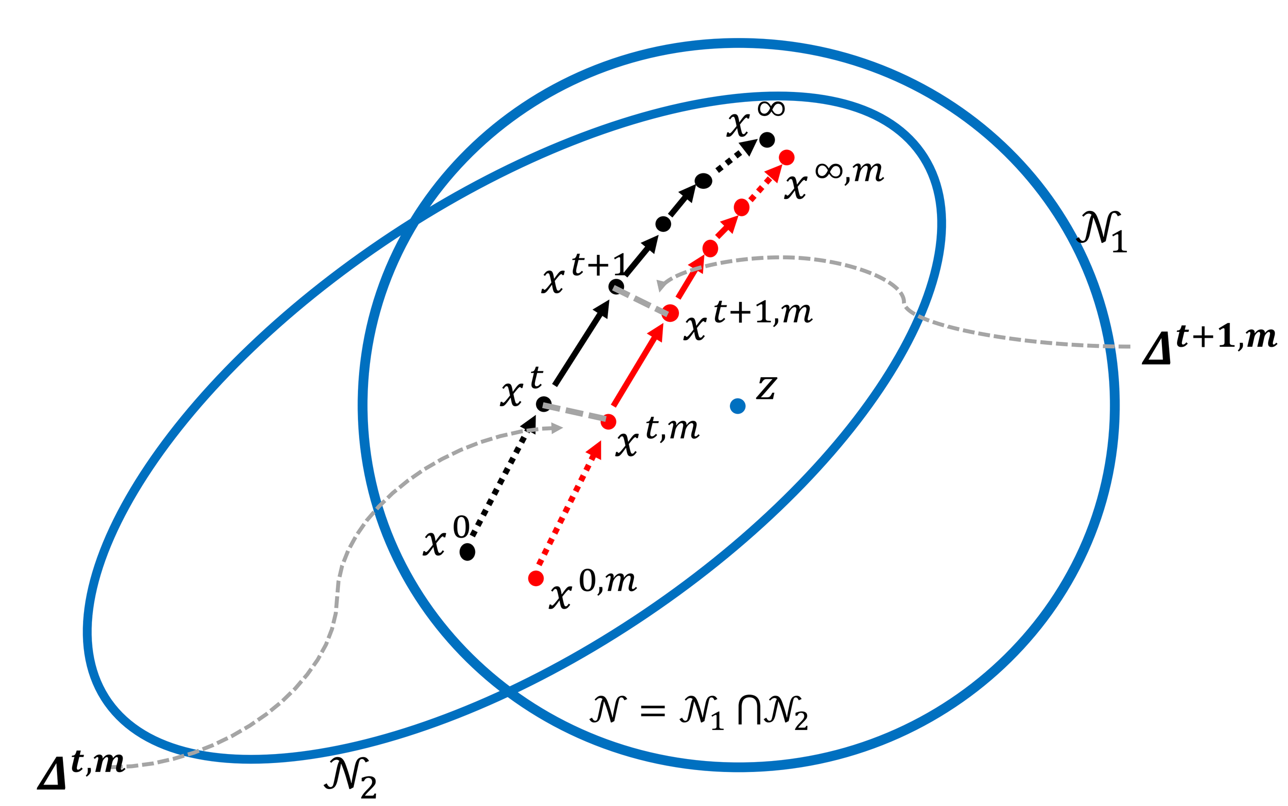
> If $\sigma = \mathcal{O}(\sqrt{n/\log n})$, with high probability for large n,
> $$\|\hat{x} - z\|_2 = \mathcal{O}(\sigma), \text{ and}$$
> $$\|\hat{x} - z\|_\infty = \mathcal{O}(\sigma\sqrt{\log n / n}).$$

- results are optimal (except for a log term);
- advantages of iterative algorithm;
- strong uniform statistical guarantee (ℓ_∞ bound).

▲ **Key idea:** introduce additional n decoupling sequences (or, leave-one-out sequences) only for analysis. For each $m \in [n]$, define $C^{(m)} := zz^* + \sigma W^{(m)}$, with

$$W^{(m)}_{k\ell} = W_{k\ell} \mathbf{1}_{\{k \ne m\}} \mathbf{1}_{\{\ell \ne m\}}, \quad x^{0,m} := \text{leading eigenvector of } C^{(m)}$$

Define GPM operator: $(\mathcal{T}x)_k = \dfrac{(Cx)_k}{|(Cx)_k|}$. Similarly, $(\mathcal{T}^{(m)}x)_k := \dfrac{(C^{(m)}x)_k}{|(C^{(m)}x)_k|}$.



$$\mathcal{N}_1 = \{x \in \mathbb{C}^n : \|Wx\|_\infty \le \kappa_2 \sqrt{n \log n}\}, \quad \mathcal{N}_2 = \{x \in \mathbb{C}^n : d_2(x, z) \le \kappa_3 \sqrt{n}\}.$$

Why decoupling?

Key: The m-th sequence $\{x^{t,m}\}_{t=0}^\infty$ is independent of $\{W_{mk}\}_{k=1}^n$ (measurements related to m-th signal) guaranteed by construction. Then, we establish

- all iterates lie in contraction region $\mathcal{N}$;
- $\Delta^{t+1,m} \le \rho \Delta^{t,m}$ + small discrepancy error, where $\rho < 1$.
- done by induction.

Contraction mapping theorem idea ⇒ convergence ✓

All $x^t \in \mathcal{N}$ ⇒ ℓ_∞ error + dual feasibility (certificate optimality) ✓

Spectral Initialization: same idea works for eigenvector initializer x^0 (with similar guarantees) → sharp ℓ_∞ bounds.

> If $\sigma = \mathcal{O}(\sqrt{n/\log n})$, then, w.h.p. for large n,
> $$\|x^0 - z\|_2 = \mathcal{O}(\sigma), \text{ and}$$
> $$\|x^0 - z\|_\infty = \mathcal{O}(\sigma\sqrt{\log n / n}).$$

Motivates analyses for other problems:

- vanilla spectral algorithm achieves exact recovery in SBM. [3]
- sharp entrywise bounds for matrix completion. [3]
- high-dimensional factor models. [4]
- and more...

References:

[1] Zhong, Yiqiao, and Nicolas Boumal. 'Near-Optimal Bounds for Phase Synchronization.' SIAM Journal on Optimization 28.2 (2018): 989-1016. (Received 2018 *SIAM Student Paper Prize*)

[2] Bandeira, Afonso S., Nicolas Boumal, and Amit Singer. 'Tightness of the maximum likelihood semidefinite relaxation for angular synchronization.' Mathematical Programming 163.1-2 (2017): 145-167.

[3] Abbe, Emmanuel, Jianqing Fan, Kaizheng Wang, and Yiqiao Zhong. 'Entrywise Eigenvector Analysis of Random Matrices with Low Expected Rank.' arXiv preprint arXiv:1709.09565 (2017).

[4] Fan, Jianqing, Weichen Wang, and Yiqiao Zhong. 'An ℓ_∞ Eigenvector Perturbation Bound and Its Application to Robust Covariance Estimation.' Journal of Machine Learning Research (to appear).