Discussion 4

- **Exponential families**
  - If there exist real-valued functions $\eta(\theta)$, $B(\theta)$ on the parameter space Θ, and real functions T and h on R^q, such that the density(frequency) functions $p(x,\theta)$ of the P_θ may be written
  $$p(x,\theta) = h(x)\exp\left[\eta(\theta)T(x) - B(\theta)\right],$$
  The family of distributions P_θ is said to be an *one-parameter exponential family.*
    * $T(X)$ is sufficient for θ.
    * η, T and B are not unique.
  - **(Theorem 1.6.2)** If the density(or frequency) function of X has the canonical form
  $$q(x,\eta) = h(x)\exp\left[\eta T(x) - A(\eta)\right]$$
  and η is an interior point of the parameter space, then the MGF of $T(X)$ is given by
  $$M(s) = \exp\left[A(s+\eta) - A(\eta)\right]$$
  for s in some neighborhood of 0. Moreover,
  $$E(T(X)) = A'(\eta), \quad \mathrm{Var}(T(X)) = A''(\eta).$$
  - **(Theorem 1.6.3** and **Corollary 1.6.1)** The multiparameter version of Theorem 1.6.2.
  - An exponential family is of rank k if and only if the generating statistic T is k-dimensional and $\{1, T_1(X), \cdots, T_k(X)\}$ are linearly independent with positive probability.
  - **(Theorem 1.6.4)** Suppose $\mathcal{P} = \{q(x,\eta) : \eta \in \varepsilon\}$ is a canonical exponential family generated by (T,h) and ε is open, then the following are equivalent:
    (i) $\mathcal{P}$ is of rank k.
    (ii) η is a parameter(identifiable).
    (iii) $Var_\eta(T)$ is positive definite.
    (iv) $\eta \rightarrow \dot{A}(\eta)$ is 1-1 on ε.
    (v) A is strictly convex on ε.

1.6.11

Obtain moment-generating functions for the sufficient statistics when sampling from the following distributions.

(b) gamma, $\Gamma(p,\lambda)$, $\theta=\lambda$, p fixed

1.6.18

Suppose $Y_1,\cdots,Y_n$ are independent with $Y_i \sim N(\beta_1+\beta_2 z_i,\sigma^2)$, where $z_1,\cdots,z_n$ are known co-variate values not all equal. Show that the family has rank 3. Give the mean vector and variance matrix of T.

1.6.31

Conjugate Normal Mixture Distributions. A Hierarchical Bayesian Normal Model. Let $\{(\mu_j,\tau_j): 1<j<k\}$ be a given collection of pairs with $\mu_j\in R, \tau_j>0$. Let (μ,σ) a random pair with $P\{(\mu,\sigma)=(\mu_j,\tau_j)\}=\lambda_j$, $0<\lambda_j<1$, $\sum_{j=1}^k\lambda_j=1$. Let θ be a random variable whose conditional distribution given $(\mu,\sigma)=(\mu_j,\tau_j)$ is $N(\mu_j,\tau_j^2)$. Consider the model $X=\theta+\epsilon$, where θ and ϵ are independent and $\epsilon\sim N(0,\sigma_0^2)$, σ_0^2 known. Not that θ has the prior density

$$\pi(\theta)=\sum_{j=1}^k \lambda_j\phi_{\tau_j}(\theta-\mu_j) \tag{1}$$

where ϕ_τ denote the $N(0,\tau^2)$ density. Also note that $(X|\theta)$ has the $N(\theta,\sigma_0^2)$ distribution.

1. Find the posterior

   $$\pi(\theta|x)=\sum_{j=1}^k P\{(\mu,\sigma)=(\mu_j,\tau_j)|x\}\pi(\theta|(\mu_j,\tau_j),x)$$

   and write it in the form

   $$\sum_{j=1}^k \lambda_j(x)\phi_{\tau_j(x)}(\theta-\mu_j(x))$$

   for appropriate $\lambda_j(x)$, $\tau_j(x)$ and $\mu_j(x)$. This shows that (1) defines a conjugate prior for the $N(\theta,\sigma_0^2)$ distribution.

2. Let $X_i=\theta+\epsilon_i$, $1<i<n$, where θ is as previously and $\epsilon_1,\cdots,\epsilon_n$ are i.i.d. $N(0,\sigma_0^2)$. Find the posterior $\pi(\theta|x_1,\cdots,x_n)$, and show that it belongs to class (1).

   Hint: Consider the sufficient statistic for $p(x|\theta)$.