STAT610 - HWK Solution 1

1.1.1 (a) $\log X_i \sim$ iid $N(\mu, \sigma^2)$, where X_i is the diameter of ith pebble $(i = 1, \cdots, n)$.
Let $\mathbf{X} = (X_1, \cdots, X_n)$.

$$f(\log \mathbf{x}) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(\log x_i - \mu)^2}{2\sigma^2} \right\} \right]$$

This is a parametric model with parameter $\theta = (\mu, \sigma^2)$ and parameter space $\Theta = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$.

(b) $X_i = \mu + 0.1 + \epsilon$, where $\epsilon \sim$ iid $N(0, \sigma^2)$ and σ^2 is known.

$$f(\mathbf{x}) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - (\mu + 0.1))^2 \right\} \right]$$

Parametric model with parameter $\theta = \mu$ and parameter space $\Theta = \mathbb{R}$.

(c) $X_i = \mu + \delta + \epsilon$, where δ is unknown positive bias .

$$f(\mathbf{x}) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - (\mu + \delta))^2 \right\} \right]$$

Parametric model with parameters $\theta = (\mu, \delta)$ and parameter space $\Theta = \{(\mu, \delta) : -\infty < \mu < \infty, \delta > 0\}$.

(d) $X_i \sim$ iid Poisson(λ), where X_i is # of eggs of ith insect $(i = 1, \cdots, n)$.
$Y_i | X_i \sim$ independent Binomial(x_i, p), where Y_i is # of eggs hatching among X_i eggs $(i = 1, \cdots, n)$.

$$f(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} f(x_i) f(y_i | x_i) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \binom{x_i}{y_i} p^{y_i} (1-p)^{x_i - y_i}$$

Parametric model with parameter $\theta = (\lambda, p)$ and parameter space $\Theta = \{(\lambda, p) : \lambda > 0, 0 \leq p \leq 1\}$.

1.1.2 (a) Unidentifiable.
Consider $\theta_1 = (\mu, \delta)$ and $\theta_2 = (\mu - 1, \delta + 1)$. Then $\theta_1 \neq \theta_2$, but $P_{\theta_1} = P_{\theta_2}$. So the model is not identifiable.

(b) Identifiable.
If $\theta_1 = (\lambda_1, p_1)$ and $\theta_2 = (\lambda_2, p_2)$ such that $f_{\theta_1}(x, y) = f_{\theta_2}(x, y)$.
Let $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = 0$, then

$$f_{\theta_1}(0, 0) = f_{\theta_2}(0, 0) \Rightarrow e^{-n\lambda_1} = e^{-n\lambda_2} \Rightarrow \lambda_1 = \lambda_2$$

Let $x_1 = \cdots = x_n = 1$ and $y_1 = \cdots = y_n = 1$, then

$$f_{\theta_1}(1, 1) = f_{\theta_2}(1, 1) \Rightarrow e^{-n\lambda_1} (\lambda_1 p_1)^n = e^{-n\lambda_2} (\lambda_2 p_2)^n \Rightarrow p_1 = p_2 \; (\because \lambda_1 = \lambda_2)$$

It proves that $\theta_1 \neq \theta_2 \Rightarrow P_{\theta_1} \neq P_{\theta_2}$, so the model is identifiable.

(c) Unidentifiable. The marginal distribution of Y_i can be obtained by

$$f_Y(y_i) = \sum_{x_i=y_i}^{\infty} f_{X,Y}(x_i, y_i) = \frac{e^{-\lambda}(\lambda p)^{y_i}}{y_i!} \sum_{x_i=y_i}^{\infty} \frac{\{\lambda(1-p)\}^{x_i-y_i}}{(x_i-y_i)!} = \frac{e^{-\lambda}(\lambda p)^{y_i}}{y_i!} e^{\lambda(1-p)} = \frac{e^{-\lambda p}(\lambda p)^{y_i}}{y_i!}$$

$\Rightarrow y_i \sim$ iid Poisson(λp)

Consider $\theta_1 = (\lambda, p)$ and $\theta_2 = (\lambda/2, 2p)$, then $\theta_1 \neq \theta_2$ but $f_{\theta_1}(y) = f_{\theta_2}(y)$. Therefore it is not identifiable.

1.1.3 (a) Unidentifiable.

$$f(x) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - (\alpha_i + \nu))^2}{2\sigma^2}\right\}$$

Consider $\theta_1 = (\alpha_1, \cdots, \alpha_p, \nu, \sigma^2)$ and $\theta_2 = (\alpha_1 + 1, \cdots, \alpha_p + 1, \nu - 1, \sigma^2)$, then $\theta_1 \neq \theta_2$ but $f_{\theta_1}(x) = f_{\theta_2}(x)$.

(b) Identifiable.
If there are $\theta = (\alpha_1, \cdots, \alpha_p, \nu, \sigma^2)$ and $\theta^* = (\alpha_1^*, \cdots, \alpha_p^*, \nu^*, \sigma^{2*})$ such that $f_\theta(x) = f_{\theta^*}(x)$, then

$$\begin{aligned}
&\alpha_i + \nu = \alpha_i^* + \nu^* (i = 1, \cdots, p) \text{ and } \sigma^2 = \sigma^{2*} \\
\Rightarrow &\sum_{i=1}^{p} \alpha_i + p\nu = \sum_{i=1}^{p} \alpha_i^* + p\nu^* \\
\Rightarrow &\nu = \nu^* \quad (\because \sum_{i=1}^{p} \alpha_i = \sum_{i=1}^{p} \alpha_i^* = 0) \\
\Rightarrow &\alpha_i = \alpha_i^* \quad (i = 1, \cdots, p)
\end{aligned}$$

(c) Unidentifiable.
$Y - X \sim N(\mu_2 - \mu_1, 2\sigma^2)$. Consider $\theta = (\mu_1, \mu_2)$ and $\theta_2 = (\mu_1 + 1, \mu_2 + 1)$, then $\theta_1 \neq \theta_2$ but $P_\theta(y - x) = P_{\theta_2}(y - x)$.

(d) Unidentifiable.
Consider $\theta_1 = (\alpha_1, \cdots, \alpha_p, \lambda_1, \cdots, \lambda_b, \nu, \sigma^2)$ and $\theta_2 = (\alpha_1 + 1, \cdots, \alpha_p + 1, \lambda_1, \cdots, \lambda_b, \nu - 1, \sigma^2)$, then $\theta_1 \neq \theta_2$ but $f_{\theta_1}(x) = f_{\theta_2}(x)$.

(e) Identifiable.
If $\theta = (\alpha_1, \cdots, \alpha_p, \lambda_1, \cdots, \lambda_b, \nu, \sigma^2)$ and $\theta^* = (\alpha_1^*, \cdots, \alpha_p^*, \lambda_1^*, \cdots, \lambda_b^*, \nu^*, \sigma^{2*})$ such that $f_\theta(x) = f_{\theta^*}(x)$, then

$$\begin{aligned}
&\alpha_i + \lambda_j + \nu = \alpha_i^* + \lambda_j^* + \nu^*, \sigma^2 = \sigma^{2*} \quad (i = 1, \cdots, p\,;\ j = 1, \cdots, b) \\
\Rightarrow &\sum_{i=1}^{p}\sum_{j=1}^{b}(\alpha_i + \lambda_j + \nu) = \sum_{i=1}^{p}\sum_{j=1}^{b}(\alpha_i^* + \lambda_j^* + \nu^*) \\
\Rightarrow &\nu = \nu^* \quad (\because \sum_{i=1}^{p} \alpha_i = \sum_{i=1}^{p} \alpha_i^* = \sum_{j=1}^{b} \lambda_j = \sum_{j=1}^{b} \lambda_j^* = 0)
\end{aligned}$$

Similarly

$$\sum_{j=1}^{b}(\alpha_i + \lambda_j + \nu) = \sum_{j=1}^{b}(\alpha_i^* + \lambda_j^* + \nu^*) \Rightarrow \alpha_i = \alpha_i^* \quad (j = 1, \cdots, b)$$

$$\sum_{i=1}^{p}(\alpha_i + \lambda_j + \nu) = \sum_{i=1}^{p}(\alpha_i^* + \lambda_j^* + \nu^*) \Rightarrow \lambda_j = \lambda_j^* \quad (i = 1, \cdots, p)$$

1.1.6 (a) Regular. P_θ are continuous with density $p(x, \theta) = \frac{1}{\theta} I_{(0,\theta)}(x)$.

(b) Not regular. $\{x; p(x) > 0\} = \{0, 1, 2, \cdots, \theta\}$ depends on θ.

(c) Not regular. P_θ are neither continuous nor discrete.

(d) Not regular. P_θ are discrete, but $\{x; p(x, \theta) > 0\} = \{0.1 + \theta, \cdots, 0.9 + \theta\}$ is dependent on θ.

1.1.7 - Continuous case

$$\begin{aligned}
&Pr(Y - c \leq t) = Pr(-Y + c \leq t) \\
\Leftrightarrow &Pr(Y \leq c + t) = 1 - Pr(Y < c - t) \\
\Leftrightarrow &F_Y(c + t) = 1 - F_Y(c - t) \\
\Leftrightarrow &\frac{dF_Y(c + t)}{dt} = \frac{-dF_Y(c - t)}{dt} \\
\Leftrightarrow &p(c + t) = p(c - t)
\end{aligned}$$

- Discrete case

$$\begin{aligned}
&Pr(Y - c = t) = Pr(-Y + c = t) \\
\Leftrightarrow &Pr(Y = c + t) = Pr(Y = c - t) \\
\Leftrightarrow &p(c + t) = p(c - t)
\end{aligned}$$