STAT610 - HWK Solution 2

1.1.14 For the median,

$$0.5 = F(\nu, \theta) = 1 - \left(\frac{\nu}{c}\right)^{-\theta} \Rightarrow \nu = c2^{1/\theta} \quad (\nu > c).$$

For the mean,

$$\mu = \int_c^\infty x dF(x) = \int_c^\infty x \left\{ \frac{\theta}{c} \left(\frac{x}{c}\right)^{-\theta-1} \right\} dx = \begin{cases} +\infty & if 0 < \theta \le 1 \\ \dfrac{c\theta}{\theta - 1} & if \theta > 1 \end{cases}$$

When $\theta > 1$, the mean exists and $\mu - \nu = \frac{c\theta}{\theta-1} - c2^{1/\theta}$. If $\theta \searrow 1$, then $\mu - \nu \to \infty$.

1.2.1 (a) The posterior probability of θ_1 given $X = 0$ can be obtained by

$$\pi(\theta_1 | X = 0) = \frac{\pi(\theta_1)p(X = 0|\theta = \theta_1)}{\pi(\theta_1)p(X = 0|\theta = \theta_1) + \pi(\theta_2)p(X = 0|\theta = \theta_2)} = \frac{0.8 \times \frac{1}{2}}{0.8 \times \frac{1}{2} + 0.4 \times \frac{1}{2}} = \frac{2}{3}$$

Similarly, $\pi(\theta_2|X = 0) = 1/3$, $\pi(\theta_1|X = 1) = 1/4$ and $\pi(\theta_2|X = 1) = 3/4$.

(b) Let $K = \sum_{i=1}^n X_i$ and $p_1 = Pr\{X = 1|\theta = \theta_1\} = 0.2$, $p_2 = Pr\{X = 1|\theta = \theta_2\} = 0.6$. Then the joint density of $(X_1, \cdots, X_n)$ can be expressed as

$$f(x_1, \cdots, x_n|\theta = \theta_j) = \prod_{i=1}^n p_j^{x_i}(1 - p_j)^{1-x_i} = p_j^k(1 - p_j)^{n-k} \quad j = 1, 2.$$

$$\begin{aligned} \therefore \pi(\theta_j|x_1, \cdots, x_n) &= \frac{\pi(\theta_1)f(x_1, \cdots, x_n|\theta_j)}{\pi(\theta_1)f(x_1, \cdots, x_n|\theta_1) + \pi(\theta_2)f(x_1, \cdots, x_n|\theta_2)} \qquad (1) \\ &= \frac{\frac{1}{2}p_j^k(1 - p_j)^{n-k}}{\frac{1}{2}p_1^k(1 - p_1)^{n-k} + \frac{1}{2}p_2^k(1 - p_2)^{n-k}}, \quad j = 1, 2 \\ &= \begin{cases} \frac{(0.2)^k(0.8)^{n-k}}{(0.2)^k(0.8)^{n-k}+(0.6)^k(0.4)^{n-k}} & j = 1 \\ \frac{(0.6)^k(0.4)^{n-k}}{(0.2)^k(0.8)^{n-k}+(0.6)^k(0.4)^{n-k}} & j = 2 \end{cases} \end{aligned}$$

We can see that $\pi(\theta|x_1, \cdots, x_n)$ only depends on $k = \sum_{i=1}^n x_i$.

(c) Plug the prior $\pi_1(\theta_1) = .25$, $\pi_1(\theta_2) = .75$ into Eq (1),

$$\begin{aligned} \pi(\theta_j|x_1, \cdots, x_n) &= \frac{\pi_1(\theta_j)p_j^k(1 - p_j)^{n-k}}{\pi_1(\theta_1)p_1^k(1 - p_1)^{n-k} + \pi_1(\theta_2)p_2^k(1 - p_2)^{n-k}}, \quad j = 1, 2 \\ &= \begin{cases} \frac{\frac{1}{4}(0.2)^k(0.8)^{n-k}}{\frac{1}{4}(0.2)^k(0.8)^{n-k}+\frac{3}{4}(0.6)^k(0.4)^{n-k}} & j = 1 \\ \frac{\frac{3}{4}(0.6)^k(0.4)^{n-k}}{\frac{1}{4}(0.2)^k(0.8)^{n-k}+\frac{3}{4}(0.6)^k(0.4)^{n-k}} & j = 2 \end{cases} \end{aligned}$$

(d) Since $K|\theta_j \sim \text{Bin}(n, \theta_j)$, the density of K is $f(k|\theta_j) = \binom{n}{k}p_j^k p_j^{n-k} \quad j = 1, 2.$

- When prior is π,

$$\begin{aligned} \pi(\theta_1|K = n/2) &= \frac{\binom{n}{n/2}p_1^{n/2}(1 - p_1)^{n/2}}{\binom{n}{n/2}p_1^{n/2}(1 - p_1)^{n/2} + \binom{n}{n/2}p_2^{n/2}(1 - p_2)^{n/2}} = \frac{(0.16)^{n/2}}{(0.16)^{n/2} + (0.24)^{n/2}} \\ &= \begin{cases} 0.4 & n = 2 \\ 1.568e - 09 \approx 0 & n = 100 \end{cases} \end{aligned}$$

- When prior is π_1, similarly,

$$\pi(\theta_1|K=n/2) = \frac{(0.16)^{n/2}}{(0.16)^{n/2} + 3\cdot(0.24)^{n/2}} = \begin{cases} 0.1818 & n=2 \\ 5.228e-10 \approx 0 & n=100 \end{cases}$$

(e) $\hat{\theta}|_\pi = \arg\max_\theta \pi(\theta|K=k) = \begin{cases} \theta_1 & \pi(\theta_1|K=k) > \pi(\theta_2|K=k) \\ \theta_2 & \pi(\theta_2|K=k) > \pi(\theta_1|K=k) \end{cases}$
where $\pi(\theta_1|K=k) > \pi(\theta_2|K=k)$ is equivalent to

$$\begin{aligned} & \pi(\theta_1)p_1^k(1-p_1)^{n-k} > \pi(\theta_2)p_2^k(1-p_2)^{n-k} \\ \Leftrightarrow\ & (0.2)^k(0.8)^{n-k} > (0.6)^k(0.4)^{n-k} \\ \Leftrightarrow\ & k\log(0.2) + (n-k)\log(0.8) > k\log(0.6) + (n-k)\log(0.4) \\ \Leftrightarrow\ & k\log(6) < n\log(2) \quad \Leftrightarrow \quad k < n\log(2)/\log(6) \end{aligned}$$

Therefore

$$\hat{\theta}|_\pi = \begin{cases} \theta_1 & k < n\log(2)/\log(6) \\ \theta_2 & k > n\log(2)/\log(6) \end{cases}$$

Similarly,

$$\hat{\theta}|_{\pi_1} = \begin{cases} \theta_1 & k < (n\log(2) - \log(3))/\log(6) \\ \theta_2 & k > (n\log(2) - \log(3))/\log(6) \end{cases}$$

- When $n=2$, $\hat{\theta}|_\pi = \hat{\theta}|_{\pi_1} = \begin{cases} \theta_1 & k=0 \\ \theta_2 & k=1,2 \end{cases}$

- When $n=100$, $\hat{\theta}|_\pi = \hat{\theta}|_{\pi_1} = \begin{cases} \theta_1 & k \le 38 \\ \theta_2 & k > 38 \end{cases}$

(f) If $\hat{\theta}|_\pi \neq \hat{\theta}|_{\pi_1}$, either $\{\hat{\theta}|_\pi = \theta_1, \hat{\theta}|_{\pi_1} = \theta_2\}$ or $\{\hat{\theta}|_\pi = \theta_2, \hat{\theta}|_{\pi_1} = \theta_1\}$.

$$\begin{aligned} \{\hat{\theta}|_\pi \neq \hat{\theta}|_{\pi_1}\} &= \{\hat{\theta}|_\pi = \theta_1, \hat{\theta}|_{\pi_1} = \theta_2\} \cup \{\hat{\theta}|_\pi = \theta_2, \hat{\theta}|_{\pi_1} = \theta_1\} \\ &= \left\{k < \frac{n\log(2)}{\log(6)} \text{ and } k > \frac{n\log(2)-\log(3)}{\log(6)}\right\} \cup \emptyset \\ &= \left\{\frac{\log 2 - (1/n)\log 3}{\log 6} < \bar{X} = \frac{K}{n} < \frac{\log 2}{\log 6}\right\} \end{aligned}$$

By CLT, $Z = \sqrt{n}(\bar{X} - \mu) \to N(0, \sigma^2)$ for some σ^2. Then,

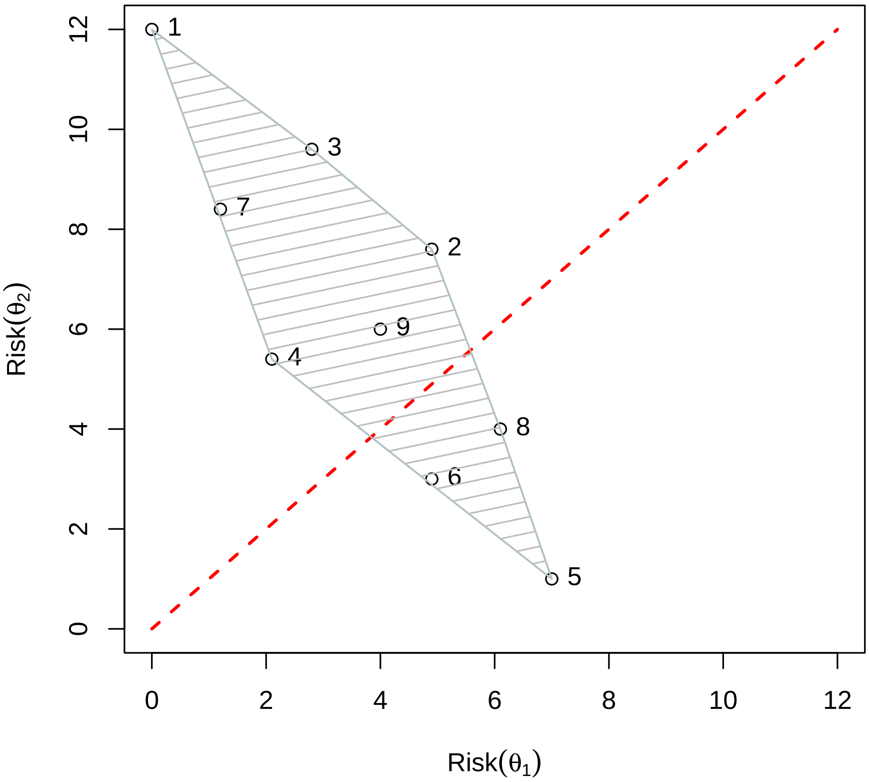
$$\Pr\left\{\sqrt{n}\left(\frac{\log 2 - (1/n)\log 3}{\log 6} - \mu\right) < Z < \sqrt{n}\left(\frac{\log 2}{\log 6} - \mu\right)\right\} \le \frac{\log 3}{\log 6\sqrt{n}}\frac{1}{\sqrt{2\pi\sigma^2}} \to 0 \quad (2)$$

as $n \to \infty$. It does not matter which prior, π or π_1, is used, since $p(x)$ will only affect μ and σ^2, but the probability in (2) still goes to 0 as $n \to \infty$.

1.3.2 (a) The risk are given in following table and figure.

Risk	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9
R(θ_1,δ)	0	4.9	2.8	2.1	7	4.9	1.2	6.1	4
R(θ_2,δ)	12	7.6	9.6	5.4	1	3	8.4	4	6

(b) δ_6



(c) From the figure above, minimax rule among the randomized rules should be

$$\delta = \begin{cases} \delta_4 & \text{w.p.} \quad \lambda \\ \delta_5 & \text{w.p.} \quad 1-\lambda \end{cases}$$

where λ satisfies $R(\theta_1, \lambda) = R(\theta_2, \lambda)$.

$$\therefore 2.1\lambda + 7(1-\lambda) = 5.4\lambda + (1-\lambda) \Rightarrow \lambda = 0.645$$

(d) From the following table, if $r = 0.5$, δ_4 is Bayes rule. If $r = 0.1$, δ_5 is Bayes rule.

Risk	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9
$r = 0.5$	6	6.25	6.2	<u>3.75</u>	4	3.95	4.8	5.05	5
$r = 0.1$	10.8	7.33	8.92	5.07	<u>1.6</u>	3.19	7.68	4.21	5.8

1.3.9 $X \sim \text{Binomial}(n, \theta_0)$, $\hat{p} = X/n$

$$MSE(\hat{p}) = \text{Bias}(\hat{p})^2 + Var(\hat{p}) = 0 + \theta_0(1-\theta_0)/n$$

$$MSE(\hat{\theta}) = \text{Bias}(\hat{p})^2 + Var(\hat{p}) = (0.02 - 0.2\theta_0)^2 + 0.64\theta_0(1-\theta_0)/n$$

$$\frac{MSE(\hat{\theta})}{MSE(\hat{p})} < 1 \Rightarrow \frac{n + 45 - 9\sqrt{n+25}}{10n+90} < \theta_0 < \frac{n + 45 + 9\sqrt{n+25}}{10n+90}$$

When $n = 25$, $0.0187 < \theta_0 < 0.393$.
When $n = 100$, $0.0407 < \theta_0 < 0.225$.