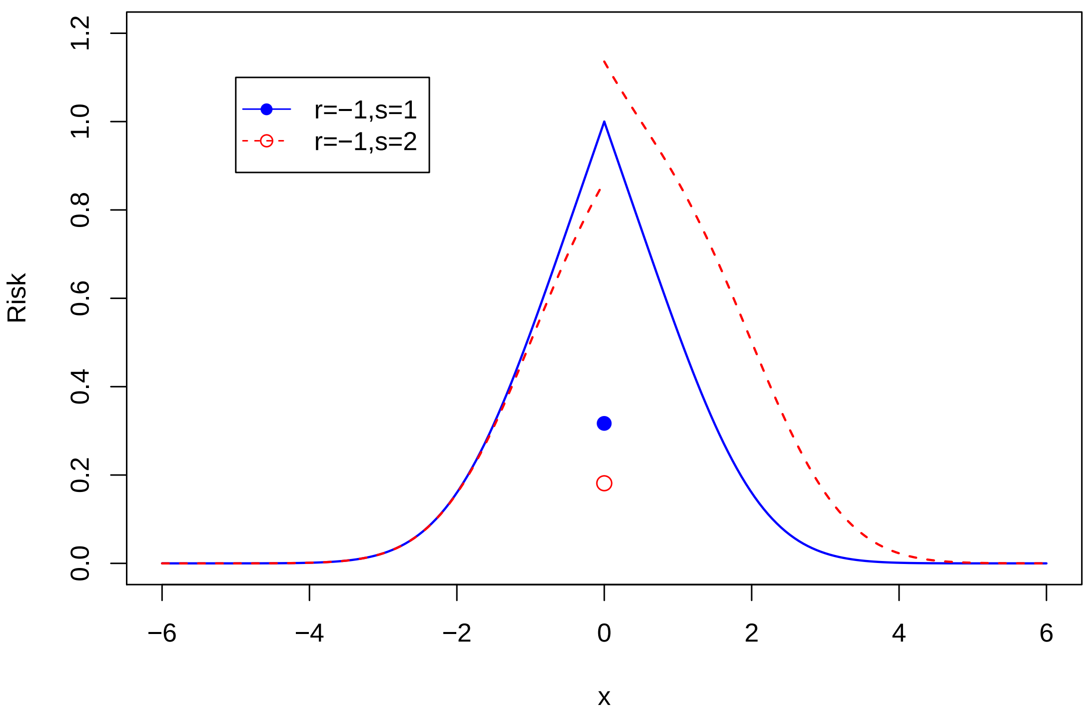
STAT610 - HWK Solution 3

1.3.3 (a) $\bar{X} \sim N(\theta, \frac{1}{n}) \Rightarrow \sqrt{n}(\bar{X}-\theta) \sim N(0,1)$

$$\begin{aligned}\text{When } \theta < 0,\ R(\theta, \delta_{r,x}(X)) &= E[l(\theta, \delta(x))] \\ &= 0Pr(\bar{X} \le t) + cPr(r \le \bar{X} \le s) + (b+c)Pr(\bar{X} > s) \\ &= cPr(\bar{X} \ge r) + bPr(\bar{X} > s) \\ &= cPr(\sqrt{n}(\bar{X}-\theta) \ge \sqrt{n}(r-\theta)) + bPr(\sqrt{n}(\bar{X}-\theta) > \sqrt{n}(s-\theta)) \\ &= c\bar{\Phi}(\sqrt{n}(r-\theta)) + b\bar{\Phi}(\sqrt{n}(s-\theta)).\end{aligned}$$

Similarly,
When $\theta = 0,\ R(\theta, \delta_{r,x}(X)) = b\Phi(\sqrt{n}r) + b\bar{\Phi}(\sqrt{n}s)$.
When $\theta > 0,\ R(\theta, \delta_{r,x}(X)) = b\Phi(\sqrt{n}(r-\theta)) + c\Phi(\sqrt{n}(s-\theta))$.

(b) See figure below. The first case has smaller risk when $\theta > 0$.



1.4.1 (a) Joint and marginal probabilities of Y and Z are given in the following table.

$Z \backslash Y$	0	1	2	3	4	p_z
0	1/70	4/35	3/35	0	0	3/14
1	0	4/35	12/35	4/35	0	4/7
2	0	0	3/35	4/35	1/70	3/14
p_y	1/70	8/35	18/35	8/35	1/70	

Thus $E(Y) = 2$, $E(Z) = 1$, $E(Y^2) = 32/7$, $E(Z^2) = 10/7$, $E(YZ) = 16/7$.

- Best predictor: $E(Y|Z=0) = \dfrac{1(4/35) + 2(3/35)}{3/14} = \frac{4}{3}$

  Similarly, $E(Y|Z=1) = 2$ and $E(Y|Z=2) = \frac{8}{3}$.

- Best linear predictor: Since the best predictor actually has a linear form $\mu(z) = \frac{4}{3} + \frac{2}{3}Z$, it should also be the best linear predictor. To verify,

$$b_1 = \frac{\text{Cov}(Y,Z)}{\text{Var}(Z)} = \frac{E(YZ) - E(Y)E(Z)}{\text{Var}(Z)} = \frac{2}{3}$$

$$a_1 = E(Y) - bE(Z) = \frac{4}{3}$$

- Best zero intercept linear predictor: $b_0 = \frac{E(YZ)}{E(Z^2)} = \frac{8}{5}$
  So, $\mu_0(z) = b_0 Z = \frac{8}{5}Z$.

(b) - Best predictor: from (1.4.6) in the textbook

$$\begin{aligned} MSPE = E(Y - E(Y|Z))^2 &= \text{Var}(Y) - \text{Var}(E(Y|Z)) \\ &= \frac{32}{7} - 2^2 - \frac{4}{9}(\frac{10}{7} - 1^2) = \frac{8}{21} \end{aligned}$$

- Best linear predictor: same with best predictor.
- Best zero intercept linear predictor:

$$MSPE = E(Y - \frac{8}{5}Z)^2 = \frac{32}{35}$$

1.4.5 The best linear predictor of Y given Z is a constant.

$$\Rightarrow \text{Cov}(Y,Z) = 0$$

The best predictor of Y given Z predicts Y perfectly.

$$\Rightarrow E(Y|Z) = Y \Rightarrow Y = g(Z)$$

Consider $Z \sim \text{N}(0,1)$ and $Y = Z^2$. Then $E(Y|Z) = Y$ and $\text{Cov}(Y,Z) = 0$.

1.5.1 (a) $X_1, \cdots, X_n \sim \text{Poisson}(\theta) \Rightarrow \sum X_i \sim \text{Poisson}(n\theta)$.

$$\begin{aligned} Pr(X_1 = x_1, \cdots, X_n = x_n | \sum X_i = t) =& \frac{Pr(X_1 = x_1, \cdots, X_n = x_n, \sum X_i = t)}{Pr(\sum X_i = t)} \\ =& \frac{Pr(X_1 = x_1, \cdots, X_n = x_n) I(\sum x_i = t)}{Pr(\sum X_i = t)} \\ =& \frac{\prod e^{-\theta}\theta^{x_i}/x_i!}{e^{-n\theta}(n\theta)^t/t!} I(\sum x_i = t) \\ =& \frac{e^{-n\theta}\theta^t / \prod x_i!}{e^{-n\theta}(n\theta)^t/t!} I(\sum x_i = t) \\ =& \frac{t!}{n^t \prod x_i!} I(\sum x_i = t) \end{aligned}$$

which does not depend on θ. Therefore, $\sum X_i$ is a sufficient statistic for θ.

(b) $Pr(X_1 = x_1, \cdots, X_n = x_n) = \left(\prod x_i!\right)^{-1} \cdot \left(e^{-n\theta}\theta^{\sum x_i}\right) = h(x)g(\sum x_i, \theta)$
by factorization theorem, $\sum X_i$ is a sufficient statistic for θ.

1.5.3 (a) $P(x_1, \cdots, x_n) = \theta^n \left(\prod x_i\right)^{\theta - 1}$., then $\prod X_i$ is sufficient.

(b) $P(x_1, \cdots, x_n) = \left\{a^n \left(\prod x_i\right)^{a-1}\right\} \{\theta^n \exp(-\theta \sum x_i^a)\}$, then $\sum X_i^a$ is sufficient.

(c) $P(x_1, \cdots, x_n) = a^{n\theta}\theta^n / (\prod x_i)^{\theta+1}$, then $\prod X_i$ is sufficient.