STAT610 - HWK Solution 5

2.1.1 Let N_1, N_2, N_3 be the number of individuals in the three different types, respectively. Note that $N_1 + N_2 + N_3 = n$. The corresponding probabilities are $\theta^2, 2\theta(1-\theta), (1-\theta)^2$.

(a) $\hat{\theta}^2 = \frac{N_1}{n}, \quad 2\hat{\theta}(1-\hat{\theta}) = \dfrac{N_2}{n} \quad \Rightarrow \hat{\theta} = \hat{\theta}^2 + \dfrac{2\hat{\theta}(1-\hat{\theta})}{2} = \dfrac{N_1}{n} + \dfrac{N_2}{2n}$

(b) $\dfrac{\hat{\theta}}{1-\hat{\theta}} = \dfrac{\frac{N_1}{n} + \frac{N_2}{2n}}{1 - (\frac{N_1}{n} + \frac{N_2}{2n})} = \dfrac{2N_1 + N_2}{2n - 2N_1 - N_2}$

(c) $E(X) = p_3 - p_1 = (1-\theta)^2 - \theta^2 = 1 - 2\theta$. Apply the mothod of moment,

$$(1 - 2\hat{\theta}) = \bar{X} = \frac{N_3 - N_1}{n} \quad \Rightarrow \hat{\theta} = \frac{N_1}{n} + \frac{N_2}{2n} = T_3$$

2.1.3 If $X \sim \beta(\alpha_1, \alpha_2)$, $E(X) = \dfrac{\alpha_1}{\alpha_1 + \alpha_2}$, $E(X^2) = \dfrac{\alpha_1(\alpha_1 + 1)}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1)}$.

Let $\hat{\mu}_1 = (\sum X_i)/n$, $\hat{\mu}_2 = (\sum X_i^2)/n$. The method of moments estimates of (α_1, α_2) is given by

$$\begin{cases} \dfrac{\alpha_1}{\alpha_1 + \alpha_2} = \hat{\mu}_1 \\ \dfrac{\alpha_1(\alpha_1 + 1)}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1)} = \hat{\mu}_2 \end{cases} \quad \Rightarrow \quad \begin{cases} \hat{\alpha}_1 = \dfrac{\hat{\mu}_1(\hat{\mu}_1 - \hat{\mu}_2)}{\hat{\mu}_2 - \hat{\mu}_1^2} \\ \hat{\alpha}_2 = \dfrac{(1 - \hat{\mu}_1)(\hat{\mu}_1 - \hat{\mu}_2)}{\hat{\mu}_2 - \hat{\mu}_1^2} \end{cases}$$

2.1.5 $X_1, \cdots, X_n \sim$ i.i.d. Bernoulli(θ).

$$V(\theta_0, \theta) = E_{\theta_0}\psi(X, \theta) = \frac{n\theta_0}{\theta} - \frac{n(1-\theta_0)}{1-\theta} = \frac{n(\theta_0 - \theta)}{\theta(1-\theta)}$$

$V(\theta_0, \theta) = 0 \Rightarrow \theta = \theta_0$, so θ_0 is the unique solution and ψ is the estimating equation. Solving $\psi(X, \hat{\theta}) = 0$ to get $\hat{\theta} = S/n$.

2.1.17 The best linear predictor is given by

$$b_1 = \frac{\text{Cov}(Y, Z)}{\text{Var}(Z)} = \frac{E(YZ) - E(Y)E(Z)}{E(Z^2) - (EZ)^2}, \quad a_1 = E(Y) - b_1 E(Z).$$

The method of moment estimate for a_1 and b_1 can be obtained by pluging in the sample moments,

$$\hat{b}_1 = \frac{(\sum Y_i Z_i)/n - \bar{Y}\bar{Z}}{(\sum Z_i^2)/n - (\bar{Z})^2}, \quad \hat{a}_1 = \bar{Y} - \hat{b}_1 \bar{Z}.$$

2.2.1 The constrast used in this problem is

$$\rho(\theta) = \sum (Y_i - \frac{\theta}{2} t_i^2)^2.$$

$$\frac{d}{d\theta}\rho(\hat{\theta}) = 0 \quad \Rightarrow \quad -\sum Y_i t_i^2 + \frac{\hat{\theta}}{2} \sum t_i^4 = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{2 \sum Y_i t_i^2}{\sum t_i^4}$$

2.2.2 Assume $\Pr(Z_i = z_i, Y_i = y_i) = \frac{1}{n}$, $i = 1, \cdots, n$. Then,

$$E(Z) = \bar{Z}, \qquad E(Y) = \bar{Y},$$
$$\mathrm{Var}(Z) = E(Z - \bar{Z})^2 = \frac{1}{n}\sum(Z_i - \bar{Z})^2$$
$$\mathrm{Cov}(Z, Y) = E[(Z - \bar{Z})(Y - \bar{Y})] = \frac{1}{n}\sum(Z_i - \bar{Z})(Y_i - \bar{Y})$$

From Thm 1.4.3., the best linear predictor is $Y = a + bZ$, where

$$\beta_2 = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Var}(Z)} = \frac{\sum(Z_i - \bar{Z})(Y_i - \bar{Y})}{\sum(Z_i - \bar{Z})^2}, \quad \beta_1 = E(Y) - \beta_2 E(Z) = \bar{Y} - \beta_2 \bar{Z}.$$

2.2.16 (a) $\hat{\theta}$ is the MLE of θ,

$$\Rightarrow L_x(\hat{\theta}) \geq L_x(\theta^*), \quad \forall \theta \in \Theta.$$

$\eta = h(\theta)$ is 1-1,

$$\Rightarrow L_x(h^{-1}(\eta)) = p(x, \eta)$$

Then for any $\eta^* = h(\theta) \in h(\Theta)$,

$$p(x, h(\hat{\theta})) = L_x(\hat{\theta}) \geq L_x(\theta^*)) = p(x, \eta^*)$$

Therefore, $h(\hat{\theta})$ is the MLE of η.

(b) The maximum likelihood estimator for ω , if exists, is defined as

$$\hat{\omega} = \arg\sup_{\omega \in \Omega} \sup\{l(\theta, x) : \theta \in \Theta, q(\theta) = \omega\}$$

Here we treat $h(\omega) = \sup\{l(\theta, x) : \theta \in \Theta, q(\theta) = \omega\}$ as a function of ω. Since q is "onto", $\forall \omega' \in \Omega$, $\exists \theta' \in \Theta$ s.t.$q(\theta') = \omega'$. Because $\hat{\theta}$ is the maximizer of $l(\theta, x)$, $l(\hat{\theta}, x) \geq l(\theta', x)$. This means $h(\hat{\omega}) \geq h(\omega')$. Therefore, MLE for ω exists and $\hat{\omega} = q(\hat{\theta})$ is a maximizer.
Remark: when q is not "onto", for some $\omega'' \in \Omega$, we cannot find $\theta'' \in \Theta$ such that $q(\theta'') = \omega''$. Therefore, the set $\{l(\theta, x) : \theta \in \Theta, q(\theta) = \omega''\}$ is not well defined. Thus we cannot maximize the function $h(\omega)$ as ω runs over the whole set of Ω.

2.2.22 The likelihood of hypergeometric distribution is given by

$$L_x(b) = \frac{\binom{b}{x}\binom{N-b}{n-x}}{\binom{N}{n}} = c \cdot \frac{b!(N-b)!}{(b-x)!(N-b-n+x)!}$$

where c is some constant which doesn't depend on b.

$$\frac{L_x(b+1)}{L_x(b)} = \frac{\frac{(b+1)!(N-b-1)!}{(b+1-x)!(N-b-1-n+x)!}}{\frac{b!(N-b)!}{(b-x)!(N-b-n+x)!}} = \frac{(b+1)(N-b-n+x)}{(b+1-x)(N-b)}$$
$$= 1 + \frac{x(N+1) - n(b+1)}{(b+1-x)(N-b)}$$

Since $(b+1-x)(N-b) > 0$,

$$\frac{L_x(b+1)}{L_x(b)} \geq 1 \Leftrightarrow L_x(b+1) \geq L_x(b) \qquad \text{when } b \leq,$$

$$\frac{L_x(b+1)}{L_x(b)} \leq 1 \Leftrightarrow L_x(b+1) \leq L_x(b) \qquad \text{when } b \geq \frac{X(N+1)}{n} - 1,$$

This shows that likelihood function is monotone increasing when $b \leq \frac{X(N+1)}{n} - 1$ and decreasing when $b \geq \frac{X(N+1)}{n} - 1$. The global maximizer $\hat{b}$ satisfies:

$$\frac{L(\hat{b}+1;X)}{L(\hat{b};X)} \leq 1, \quad \frac{L(\hat{b};X)}{L(\hat{b}-1;X)} \geq 1 \Rightarrow \frac{X(N+1)}{n} - 1 \leq \hat{b} \leq \frac{X(N+1)}{n}.$$

Therefore,

$$\hat{b}_{MLE} = \left[\frac{X(N+1)}{n}\right] \qquad \text{if } \tfrac{X(N+1)}{n} \text{ is not an integer,}$$

$$\hat{b}_{MLE} = \frac{X(N+1)}{n} \text{ or } \frac{X(N+1)}{n} - 1 \qquad \text{if } \tfrac{X(N+1)}{n} \text{ is an integer.}$$

Note that $L(\frac{X(N+1)}{n};X) = L(\frac{X(N+1)}{n} - 1;X)$ if $\frac{X(N+1)}{n}$ is an integer.