STAT610 - HWK Solution 7

2.4.1 (a) $Z_i|Y_i \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2), \sigma_1^2(1-\rho^2)\right) \qquad \Rightarrow$

$$E(Z_i|Y_i) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2)$$
$$E(Z_i^2|Y_i) = \text{Var}(Z_i|Y_i) + [E(Z_i|Y_i)]^2 = (1-\rho^2)\sigma_1^2 + \left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2)\right]^2$$
$$E(Z_iY_i|Y_i) = Y_iE(Z_i|Y_i) = Y_i\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2)\right]$$

(b) $(Z, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) \qquad \Rightarrow$

$$E(T_1) = E(\bar{Z}) = \mu_1, \qquad E(T_2) = E(\bar{Y}) = \mu_2$$
$$E(T_3) = E\left(\frac{1}{n}\sum Z_i^2\right) = E(Z_1^2) = \sigma_1^2 + \mu_1^2, \qquad E(T_4) = E\left(\frac{1}{n}\sum Y_i^2\right) = \sigma_2^2 + \mu_2^2$$
$$E(T_5) = E\left(\frac{1}{n}\sum Z_iY_i\right) = E(Z_1Y_1) = Cov(Z_1, Y_1) + E(Z_1)E(Y_1) = \rho\sigma_1\sigma_2 + \mu_1\mu_2$$

2.4.17 By example 2.4.6, $(Z_i, Y_i) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Let $W = \frac{Z_i - \mu_1}{\sigma_1}, V = \frac{Y_i - \mu_2}{\sigma_2}$, so that

$$(W, V) \sim N(0, 0, 1, 1, 0.5), \quad W|V \sim N(V/2, 3/4)$$

The probability that $E(Y_i|Z_i)$ underpredicts Y_i is

$$\begin{aligned}
Pr(E(Y_i|Z_i) < Y_i|Y_i \geq 2) &= P\left[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(Z_i - \mu_1) < Y_i \middle| Y_i \geq 2\right] \\
&= P\left[\frac{\rho(Z_i - \mu_1)}{\sigma_1} < \frac{Y_i - \mu_2}{\sigma_2} \middle| \frac{Y_i - \mu_2}{\sigma_2} \geq \frac{2 - \mu_2}{\sigma_2}\right] \\
&= P\left[\frac{1}{2}W < V \middle| V \geq \frac{1}{2}\right] = \frac{P\left[W/2 < V, V \geq 1/2\right]}{P\left[V \geq 1/2\right]} \\
&= \frac{\int_{1/2}^{\infty}\int_{-\infty}^{2v} f(w|v)f(v)dwdv}{\int_{1/2}^{\infty} f(v)dv} = \frac{\int_{1/2}^{\infty} f(v)\left[\int_{-\infty}^{2v} f(w|v)dw\right]dv}{\int_{1/2}^{\infty} f(v)dv}
\end{aligned}$$

where

$$\int_{-\infty}^{2v} f(w|v)dw = Pr(W <= 2v|V = v) = \Phi\left(\frac{2v - v/2}{\sqrt{3/4}}\right) = \Phi(\sqrt{3}v).$$

Therefore,

$$\begin{aligned}
Pr(E(Y_i|Z_i) < Y_i|Y_i \geq 2) &= \frac{\int_{1/2}^{\infty}\Phi(\sqrt{3}v)f(v)dv}{\int_{1/2}^{\infty} f(v)dv} \\
&\geq \frac{\int_{1/2}^{\infty}\Phi(\frac{\sqrt{3}}{2})f(v)dv}{\int_{1/2}^{\infty} f(v)dv} \geq \Phi(\frac{\sqrt{3}}{2}) = 0.8068
\end{aligned}$$

2.4.18 $(Z_i, Y_i) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ where

$$\begin{aligned}
\mu_2 &= E(Y) = E(\beta_1 + \beta_2 Z + \epsilon) = \beta_1 + \beta_2\mu_1 \\
\sigma_2^2 &= \text{Var}(Y) = \text{Var}(\beta_1 + \beta_2 Z + \epsilon) = \beta_2^2\sigma_1^2 + \sigma^2 \\
\rho &= \frac{\text{Cov}(Y, Z)}{\sigma_1\sigma_2} = \frac{\text{Cov}(\beta_1 + \beta_2 Z + \epsilon, Z)}{\sigma_1\sigma_2} = \frac{\beta_2\sigma_1^2}{\sigma_1\sigma_2} = \frac{\beta_2\sigma_1}{\sqrt{\beta_2^2\sigma_1^2 + \sigma^2}}
\end{aligned}$$

Observe $S(X) = \{(Z_i, Y_i) : 1 \le i \le m\} \cup \{Y_i : m + 1 \le i \le n\}$.

Let $\mathbf{T} = (T_1, T_2, T_3, T_4, T_5)$ where

$$T_1 = \bar{Z}, T_2 = \bar{Y}, T_3 = \frac{1}{n}\sum_{i=1}^{n} Z_i^2, T_4 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2, T_5 = \frac{1}{n}\sum_{i=1}^{n} Z_i Y_i.$$

- E-step:

$$\begin{aligned}
E(Z_i|Y_i) &= \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2) \\
&= \mu_1 + \frac{\beta_2\sigma_1^2}{\beta_2^2\sigma_1^2 + \sigma^2}(Y_i - \beta_1 - \beta_2\mu_1) \\
E(Z_i^2|Y_i) &= \text{Var}(Z_i|Y_i) + [E(Z_i|Y_i)]^2 \\
&= (1 - \rho^2)\sigma_1^2 + \left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(Y_i - \mu_2)\right]^2 \\
&= \frac{\sigma^2\sigma_1^2}{\beta_2^2\sigma_1^2 + \sigma^2} + \left[\mu_1 + \frac{\beta_2\sigma_1^2}{\beta_2^2\sigma_1^2 + \sigma^2}(Y_i - \beta_1 - \beta_2\mu_1)\right]^2 \\
E(Z_i Y_i|Y_i) &= Y_i E(Z_i|Y_i) \\
&= Y_i\left[\mu_1 + \frac{\beta_2\sigma_1^2}{\beta_2^2\sigma_1^2 + \sigma^2}(Y_i - \beta_1 - \beta_2\mu_1)\right]
\end{aligned}$$

- M-step:

$$\begin{aligned}
E(T_1) &= E(\bar{Z}) = \mu_1, \qquad E(T_2) = E(\bar{Y}) = \beta_1 + \beta_2\mu_1 \\
E(T_3) &= E\left(\frac{1}{n}\sum Z_i^2\right) = E(Z^2) = \sigma_1^2 + \mu_1^2 \\
E(T_4) &= E\left(\frac{1}{n}\sum Y_i^2\right) = E(Y^2) = \beta_2^2\sigma_1^2 + \sigma^2 + (\beta_1 + \beta_2\mu_1)^2 \\
E(T_5) &= E\left(\frac{1}{n}\sum Z_i Y_i\right) = E(YZ) = \beta_2\sigma_1^2 + \mu_1(\beta_1 + \beta_2\mu_1)
\end{aligned}$$

Then,

$$\dot{A}(\theta) = E_\theta(T) = (\mu_1, \beta_1 + \beta_2\mu_1, \sigma_1^2 + \mu_1^2, \beta_2^2\sigma_1^2 + \sigma^2 + (\beta_1 + \beta_2\mu_1)^2, \beta_2\sigma_1^2 + \mu_1(\beta_1 + \beta_2\mu_1))$$

Solve the following to get θ_{new}.

$$\dot{A}(\theta_{new}) = E_{\theta_{old}}(T(X)|S(X))$$

3.2.2 $X_1, \cdots, X_n \sim \text{Bernoulli}(\theta)$, $\theta \sim \text{Beta}(\alpha, \beta)$, $\theta_0 = E(\theta) = \alpha/(\alpha+\beta)$

$$\begin{aligned}\pi(\theta|x) &\propto p(x|\theta)\pi(\theta)\\ &\propto \theta^{\sum X_i}(1-\theta)^{n-\sum X_i}\cdot\theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad (\text{Let } S = \textstyle\sum X_i)\\ &\propto \theta^{S+\alpha-1}(1-\theta)^{n-S+\beta-1} \sim \text{Beta}(S+\alpha, n+\beta-S)\end{aligned}$$

Since $l(\theta, a)$ is the quadratic loss,

$$\begin{aligned}\hat{\theta}_B = E(\theta|X) &= \frac{\alpha+S}{n+\alpha+\beta}\\ &= \frac{\alpha+\beta}{n+\alpha+\beta}\cdot\frac{\alpha}{\alpha+\beta} + \frac{n}{n+\alpha+\beta}\cdot\bar{X} = w\theta_0 + (1-w)\bar{X}.\end{aligned}$$

If $\theta \sim U(0,1) = Beta(1,1)$, $\quad \hat{\theta}_B(X) = \frac{S+1}{n+2}$.

3.2.3 - MLE

$$\hat{\theta}_{MLE} = \bar{X}, \qquad \hat{q}_{MLE} = q(\hat{\theta}_{MLE}) = \bar{X}(1-\bar{X}).$$

- Bayes

$$\begin{aligned}\hat{q}_B = E[q(\theta)|X] &= \int_0^1 \theta(1-\theta)\pi(\theta|x)d\theta\\ &= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(S+\alpha)\Gamma(n+\beta-S)}\int_0^1 \theta^{S+\alpha}(1-\theta)^{n+\beta-S}d\theta\\ &= \frac{\Gamma(n+\alpha+\beta)}{\Gamma(S+\alpha)\Gamma(n+\beta-S)}\cdot\frac{\Gamma(S+\alpha+1)\Gamma(n+\beta-S+1)}{\Gamma(n+\alpha+\beta+2)}\\ &= \frac{(S+\alpha)(n+\beta-S)}{(n+\alpha+\beta)(n+\alpha+\beta+1)}\\ q(\hat{\theta}_B) = \hat{\theta}_B(1-\hat{\theta}_B) &= \frac{(\alpha+S)(n+\beta-S)}{(n+\alpha+\beta)^2}\end{aligned}$$

$$\hat{q}_B - q(\hat{\theta}_B) = -\frac{(\alpha+S)(n+\beta-S)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)} < 0 \Rightarrow \hat{q}_B \neq q(\hat{\theta}_B)$$

3.2.4 $\lambda = \theta/(1-\theta) \Leftrightarrow \theta = \lambda/(1+\lambda)$.

$$p(x|\lambda) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} = \left(\frac{\lambda}{1+\lambda}\right)^S\left(\frac{1}{1+\lambda}\right)^{n-S} = \frac{\lambda^S}{(1+\lambda)^n}$$

$$\begin{aligned}\hat{\lambda}_B &= \frac{\int_0^\infty \lambda p(x|\lambda)\pi(\lambda)d\lambda}{\int_0^\infty p(x|\lambda)\pi(\lambda)d\lambda} = \frac{\int_0^\infty \lambda^{S+1}(1+\lambda)^{-n}d\lambda}{\int_0^\infty \lambda^S(1+\lambda)^{-n}d\lambda}\\ &= \frac{\int_0^1\left(\frac{\theta}{1-\theta}\right)^{n\bar{X}+1}\left(\frac{1}{1-\theta}\right)^{-n}\frac{1}{(1-\theta)^2}d\theta}{\int_0^1\left(\frac{\theta}{1-\theta}\right)^{n\bar{X}}\left(\frac{1}{1-\theta}\right)^{-n}\frac{1}{(1-\theta)^2}d\theta} = \frac{\int_0^1 \theta^{S+1}(1-\theta)^{n-S-3}d\theta}{\int_0^1 \theta^S(1-\theta)^{n-S-2}d\theta}\\ &= \frac{\Gamma(S+2)\Gamma(n-S-2)/\Gamma(n)}{\Gamma(S+1)\Gamma(n-S-1)/\Gamma(n)} = \frac{S+1}{n-S-2} \quad (\text{if } S < n-2)\end{aligned}$$

3.2.9 (a) Posterior risk $r(0|x) = E(l(\theta, 0)|x)$, $r(1|x) = E(l(\theta, 1)|x)$. By Proposition 3.2.1, the Bayes rule is then

$$\delta^*(X) = 0 \text{ (“Accept bioequivalence”) if } r(0|x) < r(1|x),$$

where

$$\begin{aligned} r(0|x) < r(1|x) &\Leftrightarrow E(l(\theta,0)|x) < E(l(\theta,1)|x) \\ &\Leftrightarrow E(l(\theta,0) - l(\theta,1)|x) < 0 \\ &\Leftrightarrow E(\lambda(\theta)|x) < 0. \end{aligned}$$

From Example 3.2.1, $\theta|x \sim N(\eta(n), \tau_0^2(n))$, where

$$\eta(n) = \eta_0 \left(\frac{1/\tau_0^2}{1/\tau_0^2 + n/\sigma_0^2} \right) + \bar{X} \left(\frac{n/\sigma_0^2}{1/\tau_0^2 + n/\sigma_0^2} \right), \quad \tau_0^2(n) = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_0^2} \right)^{-1}.$$

Then

$$\begin{aligned} E(\lambda(\theta)|x) &= E\left[r - \exp\left\{ -\frac{\theta^2}{2c^2} \right\} \middle| x \right] \\ &= r - \frac{1}{\sqrt{2\pi\tau_0^2(n)}} \int_{-\infty}^{\infty} \exp\left(-\frac{\theta^2}{2c^2} \right) \exp\left(-\frac{(\theta - \eta(n))^2}{2\tau_0^2(n)} \right) d\theta \\ &= r - \frac{1}{\sqrt{2\pi\tau_0^2(n)}} \exp\left(-\frac{\eta(n)^2}{2(c^2 + \tau_0^2(n))} \right) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \left(\frac{1}{c^2} + \frac{1}{\tau_0^2(n)} \right) \left(\theta - \frac{c^2\eta(n)}{c^2 + \tau_0^2(n)} \right)^2 \right] d\theta \\ &= r - \frac{1}{\sqrt{2\pi\tau_0^2(n)}} \exp\left(-\frac{\eta(n)^2}{2(c^2 + \tau_0^2(n))} \right) \sqrt{2\pi \left(\frac{1}{c^2} + \frac{1}{\tau_0^2(n)} \right)^{-1}} \\ &= r - \frac{\sqrt{c^2}}{\sqrt{c^2 + \tau_0^2(n)}} \exp\left(-\frac{\eta(n)^2}{2(c^2 + \tau_0^2(n))} \right) \end{aligned}$$

Since $r = \exp\left(-\frac{\epsilon^2}{2c^2} \right)$,

$$E(\lambda(\theta)|x) < 0 \Leftrightarrow \eta(n)^2 = \{E(\theta|x)\}^2 < (c^2 + \tau_0^2(n)) \left[\log\left(\frac{c^2}{c^2 + \tau_0^2(n)} \right) + \frac{\epsilon^2}{c^2} \right]$$

(b) If $\eta_0 = 0, \tau_0^2 \to \infty$, then

$$\eta(n) \to \bar{X}, \qquad \tau_0^2(n) \to \frac{\sigma_0^2}{n}$$

The Bayes rule becomes

$$\text{“Accept bioequivalence” if } \bar{X}^2 < \left(c^2 + \frac{\sigma_0^2}{n} \right) \left\{ \log\left(\frac{c^2}{c^2 + \sigma_0^2/n} \right) + \frac{\epsilon^2}{c^2} \right\}$$

(c) If $n \to \infty$, $\eta(n) \to \bar{X}$ and $\tau_0^2(n) \to 0$ in both (a) and (b). So the Bayes rule becomes

$$\text{“Accept bioequivalence” if } \bar{X}^2 < \epsilon^2$$