STAT610 - HWK Solution 8

3.3.5 (a) $\mu = 0$. $\theta = 1/\sigma^2 \sim \text{Gamma}(\alpha, \beta)$.

– Posterior:

$$\pi(\theta|x) \propto p(x|\theta)\pi(\theta) \propto \theta^{\frac{n}{2}} e^{-\frac{\theta}{2}\sum X_i^2} \theta^{\alpha-1} e^{-\beta\theta}$$
$$\propto \theta^{\frac{n}{2}+\alpha-1} \exp\left\{-\left(\frac{1}{2}\sum X_i^2 + \beta\right)\theta\right\}$$
$$\Rightarrow \theta|x \sim \text{Gamma}\left(\frac{n}{2}+\alpha,\ \frac{1}{2}\sum X_i^2 + \beta\right)$$

– Bayes rule: $\delta_B(X) = \arg\min E[(\frac{d}{\sigma^2} - 1)^2|X]$.

$$E\left[(d/\sigma^2 - 1)^2 \middle| X\right] = E[(d\theta - 1)^2|X]$$
$$= d^2 E(\theta^2|X) - 2dE(\theta|X) + 1$$

$$\Rightarrow \delta_B(X) = \frac{E(\theta|X)}{E(\theta^2|X)} = \frac{\frac{n/2+\alpha}{\sum X_i^2/2+\beta}}{\frac{n/2+\alpha}{(\sum X_i^2/2+\beta)^2} + (\frac{n/2+\alpha}{\sum X_i^2/2+\beta})^2} = \frac{\sum X_i^2 + 2\beta}{2\alpha+n+2}$$

– Risk of $\delta_B(X)$:

$$R(\theta, \delta_B(X)) = E\left(\frac{\sum X_i^2 + 2\beta}{2\alpha+n+2}\frac{1}{\sigma^2} - 1\right)^2$$
$$= E\left(\frac{Y + 2\beta/\sigma^2}{2\alpha+n+2} - 1\right)^2 \qquad (\text{Let } Y = \sum X_i^2/\sigma^2 \sim \chi_n^2 \sim \Gamma(n/2, 1/2))$$
$$= \frac{E((Y-n) + (2\beta/\sigma^2 - 2\alpha - 2))^2}{(2\alpha+n+2)^2}$$
$$= \frac{E(Y-n)^2 + (2\beta/\sigma^2 - 2\alpha - 2)^2}{(2\alpha+n+2)^2} \qquad (E(Y) = n)$$
$$= \frac{2n + (2\beta/\sigma^2 - 2\alpha - 2))^2}{(2\alpha+n+2)^2} \qquad (\text{Var}(Y) = 2n)$$

– Bayes risk:

$$r(\pi, \delta_B) = E(R(\theta, \delta_B)) = E\left(\frac{2n + (2\beta/\sigma^2 - 2\alpha - 2))^2}{(2\alpha+n+2)^2}\right)$$
$$= \frac{2n + E[(2\beta(\theta - \alpha/\beta) - 2)]^2}{(2\alpha+n+2)^2}$$
$$= \frac{2n + 4\beta^2\frac{\alpha}{\beta^2} + 4}{(2\alpha+n+2)^2} = \frac{2}{2\alpha+n+2}$$

– Risk of $\delta^*(X)$:

$$R(\theta, \delta^*(X)) = E\left(\frac{1}{n+2}\sum X_i^2/\sigma^2 - 1\right)^2 = E\left(\frac{Y}{n+2} - 1\right)^2$$
$$= \frac{E(Y^2)}{(n+2)^2} - \frac{2E(Y)}{n+2} + 1 = \frac{n^2+2n}{(n+2)^2} - \frac{2n}{n+2} + 1 = \frac{2}{n+2}$$

Let $\beta > 0$ and $\alpha \to 0^+$,

$$r(\pi, \delta_B) = \frac{2}{2\alpha + n + 2} \to \frac{2}{n+2} = R(\theta, \delta^*(X)),$$

By Theorem 3.3.3, $\delta^*(X)$ is minimax.

(b) The Risk of δ_c:

$$\begin{aligned} R(\theta, \delta_c) &= E(c\sum X_i^2/\sigma^2 - 1)^2 \\ &= E(cY-1)^2 = c^2E(Y^2) - 2cE(Y) + 1 \end{aligned}$$

Hence, $R(\theta, \delta_c)$ is minimized when $c = \frac{E(Y)}{E(Y^2)} = \frac{n}{n^2+2n} = \frac{1}{n+2}$, i.e. $\delta^*(X)$ is uniformly best among all rules of δ_c.
The MLE of σ^2 is given by $\delta_{MLE}(X) = \frac{1}{n}\sum X_i^2$,

$$\Rightarrow R(\theta, \delta_{MLE}(X)) > R(\theta, \delta^*(X)), \forall\theta$$

i.e. MLE is inadmissible.

(c) The Risk of δ_c:

$$\begin{aligned} R(\theta, \delta_c) &= E(c\sum(X_i - \bar{X})^2/\sigma^2 - 1)^2 \\ &= E(cZ-1)^2 = c^2E(Z^2) - 2cE(Z) + 1 \quad (Z = \sum(X_i - \bar{X})^2/\sigma^2 \sim \chi^2_{n-1} \sim \Gamma((n-1)/2, 1/2)) \end{aligned}$$

Hence, $R(\theta, \delta_c)$ is minimized when $c = \frac{E(Z)}{E(Z^2)} = \frac{n-1}{(n-1)^2+2(n-1)} = \frac{1}{n+1}$, i.e. $\delta(X) = \frac{1}{n+1}\sum(X_i - \bar{X})^2$ is uniformly best among all rules of δ_c.
For $\delta_{MLE}(X) = \frac{1}{n}\sum(X_i - \bar{X})^2$ and $S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$,

$$R(\theta, \delta_{MLE}(X)) > R(\theta, \delta(X)), \quad R(\theta, S^2) > R(\theta, \delta(X)) \quad \forall\theta,$$

So they are both inadmissible.

3.3.7 $X \sim$ Poisson(λ), $\pi(\lambda) =$ Gamma$(1, 1/k)$.

- posterior

$$\begin{aligned} \pi(\lambda|x) &\propto p(x|\lambda)\pi(\lambda) \propto \lambda^x e^{-\lambda} e^{-\lambda/k} \\ &\propto \lambda^x e^{-(\lambda(1+1/k)} \\ &\Rightarrow \lambda|x \sim \text{Gamma}(X+1,\ 1+1/k) \end{aligned}$$

- Bayes rule:

$$\begin{aligned} E\left[\frac{(\lambda-1)^2}{\lambda}|X\right] &= a^2E(1/\lambda|X] - 2a + E(\lambda|X) \\ \Rightarrow \delta_B(X) = \arg\min E\left[(\lambda-1)^2/\lambda|X\right] &= \frac{1}{E(1/\lambda|X)} \end{aligned}$$

where

$$E(1/\lambda|X) = \frac{(1+1/k)^{X+1}}{\Gamma(X+1)} \int_0^\infty \frac{1}{\lambda} \lambda^X e^{-\lambda(1+1/k)} d\lambda$$
$$= \begin{cases} \infty & X = 0 \\ \frac{1+1/k}{X} & X > 0 \end{cases}$$

$$\Rightarrow \delta_B(X) = \frac{X}{1+1/k}$$

- Risk of $\delta_B(X)$:

$$R(\lambda, \delta_B(X)) = E\left(\frac{(\lambda - \frac{x}{1+1/k})^2}{\lambda}\right) = \frac{1+\lambda/k^2}{(1+1/k)^2}$$

- Bayes risk:

$$r(\pi, \delta_B) = E(R(\theta, \delta_B)) = \frac{1+k/k^2}{(1+1/k)^2} = \frac{1}{1+1/k}$$

- Risk of X:

$$R(\theta, X) = E\left(\frac{(\lambda - X)^2}{\lambda}\right) = \text{Var}(X)/\lambda = 1$$

Let $k \to \infty$,

$$r(\pi, \delta_B) = \frac{1}{1+1/k} \to 1 = R(\theta, X),$$

By Theorem 3.3.3, $\delta = X$ is minimax.

3.4.3 $\eta = h(\theta) \Rightarrow d\eta/d\theta = h'(\theta),\ d\theta/d\eta = 1/h'(\theta) = 1/h'(h^{-1}(\theta))$

(a) the Fisher information is given by

$$I_q(\eta) = E\left[\frac{\partial}{\partial \eta} \log q(X, \eta)\right]^2 = E\left[\frac{\partial}{\partial \eta} \log p(X, \theta)\right]^2$$
$$= E\left[\frac{\partial}{\partial \theta} \log p(X, \theta) \cdot \frac{\partial \theta}{\partial \eta}\right]^2 = E\left[\frac{\partial}{\partial \theta} \log p(X, \theta)\right]^2 \cdot \left(\frac{\partial \theta}{\partial \eta}\right)^2$$
$$= I_p(h^{-1}(\eta))/[h'(h^{-1}(\eta))]^2$$

(b) $B_p(\theta) = [\psi_p'(\theta)]^2/I_p(\theta), \quad B_q(\eta) = [\psi_q'(\eta)]^2/I_q(\eta).$

$$\psi_q'(\eta) = \frac{d\psi}{d\eta} = \frac{d\psi}{d\theta}\frac{d\theta}{d\eta} = \frac{\psi_p'(\theta)}{h'(\theta)} \Rightarrow B_q(\eta) = \frac{[\psi_p'(\theta)/h'(\theta)]^2}{I_p(\theta)/[h'(\theta)]^2} = \frac{[\psi_p'(\theta)]^2}{I_p(\theta)} = B_p(h^{-1}(\eta))$$

3.4.5 (b) Similar to 3.3.5(b), $\hat{\sigma}^2 = \frac{1}{n+2}\sum(X_i - \mu_0)^2$ is uniformly best among all estimates of the form $\hat{\sigma}_c^2 = c\sum(X_i - \mu_0)^2$, including $\hat{\sigma}_0^2 = \frac{1}{n}\sum(X_i - \mu_0)^2$. Thus, $\hat{\sigma}_0^2$ is inadmissible.