STAT610 - HWK Solution 9

3.3.13 $X|\theta \sim \text{Multinomial}(n, \theta)$

- prior :

$$\pi(\theta_1, \cdots, \theta_{k-1}) = (k-1)!, \quad 0 < \theta_j < 1, \quad \sum_{j=1}^{k} \theta_j = 1$$

- posterior :

$$\pi(\theta|x) \propto p(x|\theta)\pi(\theta) \propto \theta_1^{x_1} \cdots \theta_k^{x_k}, \quad 0 < \theta_j < 1, \quad \sum_{j=1}^{k} \theta_j = 1$$

$$\theta|x \sim \text{Dirichlet}(X_1 + 1, \cdots, X_k + 1), \quad E(\theta_j|X) = \frac{X_j + 1}{n + k}.$$

- loss function : Kullback-leibler divergence

$$l_p(\theta, a) = -E_\theta \left[\log \frac{p(X, a)}{p(X, \theta)} \right] = -\sum_{i=1}^{k} E_\theta(X_i) \log \left(\frac{a_i}{\theta_i} \right) = -n \sum_{i=1}^{k} \theta_i \log \left(\frac{a_i}{\theta_i} \right)$$

- Bayes estimate

$$\begin{aligned} E[l_p(\theta, a)|X] &= -n \sum_{i=1}^{k} E(\theta_i|X) \log a_i + n \sum_{i=1}^{k} E(\theta_i \log \theta_i|X) \\ &= -n \sum_{i=1}^{k-1} E(\theta_i|X) \log a_i - nE(\theta_k|X) \log \left(1 - \sum_{i=1}^{k-1} a_i \right) + n \sum_{i=1}^{k} E(\theta_i \log \theta_i|X) \\ \frac{\partial}{\partial a_i} E[l_p(\theta, a)|X] &= -n \frac{E[\theta_i|X]}{a_i} + n \frac{E[\theta_k|X]}{1 - \sum_{i=1}^{k-1} a_i} \\ &= -n \frac{E[\theta_i|X]}{a_i} + n \frac{E[\theta_k|X]}{a_k} \quad , i = 1, \cdots, k-1 \end{aligned}$$

$$\Rightarrow a_i = a_k \frac{E[\theta_i|X]}{E[\theta_k|X]}, \quad i = 1, \cdots, k$$

Since $\sum_{i=1}^{k} a_i = 1$, $a_i = \frac{E[\theta_i|X]}{\sum_{i=1}^{k} E[\theta_i|X]} = \frac{X_i+1}{n+k} \quad , i = 1, \cdots, k-1$

$$\delta_B(X) = \left(\frac{X_1 + 1}{n + k}, \cdots, \frac{X_k + 1}{n + k} \right)$$

3.4.22 $p(x, \theta) = \frac{1}{\theta}$, $x < \theta \quad \Rightarrow \quad T = \frac{\partial}{\partial \theta} \log p(x, \theta) = -\frac{1}{\theta}$

(a) $E(T) = -1/\theta \neq 0$

(b) $Var(T) = 0, \quad I(\theta) = E(T^2) = Var(T) + ET^2 = \frac{1}{\theta^2}$

(c) $E(2X) = 2 \cdot \frac{\theta - 0}{2} = \theta, \quad Var(2X) = 4 \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3} < \frac{1}{I(\theta)} = \theta^2$

3.5.5 Random sample $X_1, \cdots, X_n \Rightarrow$ Order statistics $X_{(1)}, \cdots, X_{(n)}$

- Empirical Distribution :

$$\hat{F}(x) = \hat{P}[X \leq x] = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$$

- αth quantile of the distribution :

$$x_\alpha = \frac{1}{2}\left(\inf\{x : F(x) \geq \alpha\} + \sup\{x : F(x) \leq \alpha\}\right)$$

- αth sample quantile :

$$\hat{x}_\alpha = \frac{1}{2}\left(\inf\left\{x : \hat{F}(x) \geq \alpha\right\} + \sup\left\{x : \hat{F}(x) \leq \alpha\right\}\right)$$

If $n\alpha$ is not an integer, $\hat{x}_\alpha = x_{([n\alpha]+1)}$, $\hat{x}_{1-\alpha} = x_{(n-[n\alpha])}$.
If $n\alpha$ is an integer, $\hat{x}_\alpha = \frac{1}{2}\left\{x_{(n\alpha)} + x_{(n\alpha+1)}\right\}$, $\hat{x}_{1-\alpha} = \frac{1}{2}\left\{x_{(n-n\alpha)} + x_{(n-n\alpha+1)}\right\}$

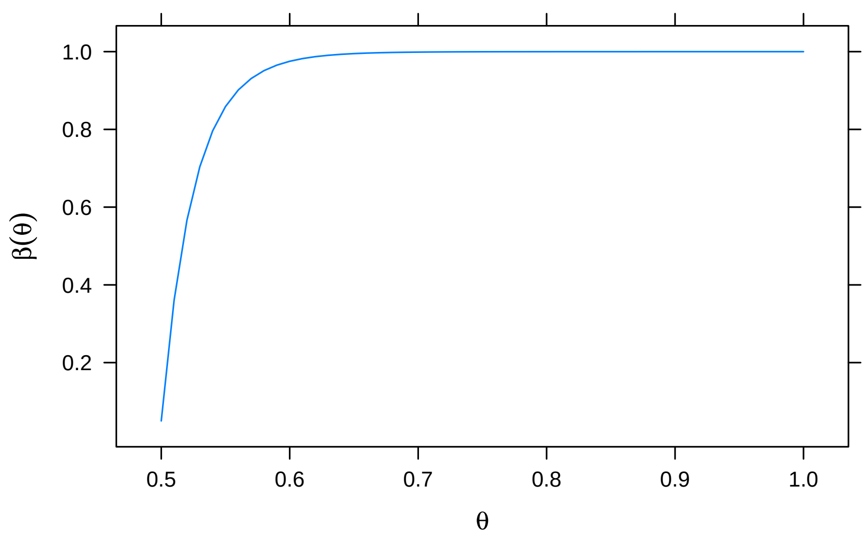
To estimate $\mu_\alpha = \dfrac{1}{1-2\alpha}\int_{x_\alpha}^{x_{1-\alpha}} x dF(x) = \dfrac{\int_{x_\alpha}^{x_{1-\alpha}} x dF(x)}{F(x_{1-\alpha}) - F(x_\alpha)}$,

$$\hat{\mu}_\alpha = \frac{\int_{\hat{x}_\alpha}^{\hat{x}_{1-\alpha}} x d\hat{F}(x)}{\hat{F}(\hat{x}_{1-\alpha}) - \hat{F}(\hat{x}_\alpha)} = \frac{\frac{1}{n}(X_{([n\alpha]+1)} + \cdots + X_{(n-[n\alpha])})}{\frac{1}{n}(n - 2[n\alpha])} = \bar{X}_\alpha$$

4.1.1 (a) $\beta(\theta) = P_\theta(\delta_c(X) = 1) = P_\theta(M_n \geq c)$.
If $\theta < c$, $\beta(\theta) = 0$.
If $\theta \geq c$, $\beta(\theta) = P_\theta(M_n \geq c) = 1 - P_\theta(X_1 < c, \cdots, X_n < c) = 1 - \left(\frac{c}{\theta}\right)^n$.

(b) $\alpha_c = 0.05 \;\Rightarrow\; \sup_{\theta \leq 1/2}\left\{1 - \left(\frac{c}{\theta}\right)^n\right\} = 1 - (2c)^n = 0.0 \;\Rightarrow\; c = \frac{1}{2}(0.95)^{\frac{1}{n}}$

(c)



(d) $\beta(3/4) = 0.98 \;\Rightarrow\; 1 - \left(\frac{0.95^{1/n}/2}{3/4}\right)^n = 0.98 \;\Rightarrow\; n = \log\frac{0.95}{1-0.98} = 9.52 \;\Rightarrow\; n \geq 10$

(e) $\alpha(c) = 1-(2c)^n \quad \Rightarrow \quad c(\alpha) = \frac{1}{2}(1-\alpha)^{1/n}$

$$\begin{aligned} p-\text{value } &= \min_\alpha\{\text{reject } H\} = \min_\alpha\{M_n \geq c(\alpha)\} = \min_\alpha\{M_n \geq \frac{1}{2}(1-\alpha)^{1/n}\} \\ &= \min_\alpha\{\alpha \geq 1-(2M_n)^n\} = 1-(2M_n)^n = 0.558 \end{aligned}$$

4.1.2 $X_1, \cdots, X_n \sim \text{Exp}(\lambda)$.

(a) $\alpha(c) = \sup_{\mu \leq \mu_0}\{P_\mu(\bar{X} \geq c)\}$ where

$$P_\mu(\bar{X} \geq c) = P_\mu(2\lambda \sum X_i \geq 2\lambda nc) = P(T \geq \frac{2nc}{\mu}) \qquad (T = 2\lambda \sum X_i \sim \chi^2_{2n})$$

Then $\alpha(c) = P(T \geq \frac{2nc}{\mu_0}) \Rightarrow \frac{2nc}{\mu_0} = \chi^2_{2n,1-\alpha} \Rightarrow c = \frac{\mu_0 \chi^2_{2n,1-\alpha}}{2n}$

(b) $\beta(\mu) = P_\mu(\bar{X} \geq \frac{\mu_0\chi^2_{2n,1-\alpha}}{2n}) = P(T \geq \frac{\mu_0}{\mu}\chi^2_{2n,1-\alpha})$

(c) $E(\bar{X}) = 1/\lambda = \mu$, $Var(\bar{X}) = \frac{1}{n}\frac{1}{\lambda^2} = \mu^2/n$. By CLT, $\frac{\bar{X}-\mu}{\mu/\sqrt{n}} \to N(0,1)$

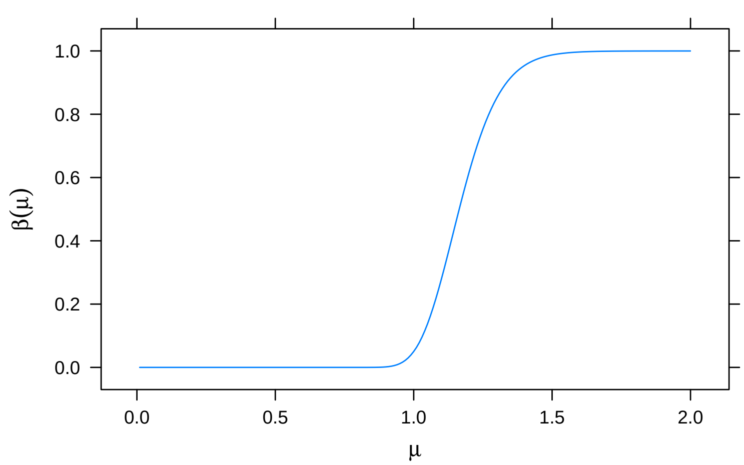
- Approximate the critical region: By CLT,

$$\frac{\chi^2_{2n} - 2n}{2n/\sqrt{n}} \to N(0,1) \quad \Rightarrow \quad \frac{\mu_0 \chi^2_{2n,1-\alpha}}{2n} \to \mu_0\left(1 + \frac{z_{1-\alpha}}{\sqrt{n}}\right)$$

- Approximate the power function

$$\begin{aligned} \beta(\mu) = P_\mu\left(\bar{X} \geq \frac{\mu_0\chi^2_{2n,1-\alpha}}{2n}\right) &\approx P_\mu\left(\bar{X} \geq \mu_0\left(1 + \frac{z_{1-\alpha}}{\sqrt{n}}\right)\right) \\ = P\left(\frac{\bar{X}-\mu}{\mu/\sqrt{n}} \geq \frac{\mu_0\left(1+\frac{z_{1-\alpha}}{\sqrt{n}}\right) - \mu}{\mu/\sqrt{n}}\right) &= P\left(\frac{\bar{X}-\mu}{\mu/\sqrt{n}} \geq \frac{\mu_0}{\mu}z_{1-\alpha} + \sqrt{n}\frac{\mu_0-\mu}{\mu}\right) \\ \approx 1 - \Phi\left(\frac{\mu_0}{\mu}z_{1-\alpha} + \sqrt{n}\frac{\mu_0-\mu}{\mu}\right) &= \Phi\left(\frac{\mu_0}{\mu}z_\alpha + \sqrt{n}\frac{\mu-\mu_0}{\mu}\right) \end{aligned}$$

- Plot of the power function when $\alpha = 0.05, n = 100, \mu_0 = 25$.



(d) $\mu_0 = 25$, $\bar{X} = 33.95$, $n = 20$.

$$\bar{X} < c = \frac{\mu_0 \chi^2_{2n,1-\alpha}}{2n} = 34.85 \qquad \text{or } \bar{X} < \mu_0(1 + z_{1-\alpha}/\sqrt{n}) = 34.20$$

So H is not rejected at level $\alpha = 0.05$.

4.1.10 (a) Let $X_i' = \frac{X_i - a}{b}$, $i = 1, \cdots, n$, then $\bar{X}' = \frac{\bar{X}-a}{b}$ and $\hat{\sigma}' = \sigma/b$.
The empirical distribution of X' is given by

$$\begin{aligned}\hat{F}'(\bar{X}' + \hat{\sigma}'x) &= \frac{1}{n}\sum I\left(X_i' \le \bar{X}' + \hat{\sigma}'x\right) = \frac{1}{n}\sum I\left(\frac{X_i - a}{b} \le \frac{\bar{X} - a}{b} + \frac{\hat{\sigma}}{b}x\right) \\ &= \frac{1}{n}\sum I(X_i \le \bar{X} + \hat{\sigma}x) = \hat{F}(\bar{X} + \hat{\sigma}x)\end{aligned}$$

$$\therefore T_n' = \sup_x |\hat{F}'(\bar{X}' + \hat{\sigma}'x) - \Phi(x)| = \sup_x |\hat{F}(\bar{X} + \hat{\sigma}x) - \Phi(x)| = T_n$$

(b) Since $T_n(X) = T_n(X')$, $L_{N(\mu,\sigma^2)}(T_n(X)) = L_{N(0,1)}(T_n(\frac{X-\mu}{\sigma})) = L_{N(0,1)}(T_n(X))$

4.1.11 (a) Assume that $X_1, \cdots, X_n \sim$ iid F_0. Let $U_i = F_0(X_i)$, thus $U_i \sim \text{Unif}(0,1)$ and

$$\hat{F}(x) = \frac{1}{n}\sum I(X_i \le x) = \frac{1}{n}\sum I(F_0(X_i) \le F_0(x)) = \hat{U}(F_0(x)),$$

where $\hat{U}$ denotes the empirical distribution function of $U_1, \cdots, U_n$. Let $u = F_0(x) \in (0,1)$,

$$\begin{aligned}S_{\psi,\alpha} &= \sup_{0<u<1} \psi(u)|\hat{U}(u) - u|^\alpha, \quad T_{\psi,\alpha} = \sup_{0<u<1} \psi(\hat{U}(u))|\hat{U}(u) - u|^\alpha \\ U_{\psi,\alpha} &= \int_0^1 \psi(u)|\hat{U}(u) - u|^\alpha du, \quad V_{\psi,\alpha} = \int_0^1 \psi(\hat{U}(u))|\hat{U}(u) - u|^\alpha d\hat{U}(u)\end{aligned}$$

Therefore, all of them do not depend on F_0.

(b) The Cramer-von Mises statistic is given by

$$\begin{aligned}V_{1,2} &= \int |\hat{F}(x) - F_0(x)|^2 d\hat{F}(x) = \frac{1}{n}\sum \left(\hat{F}(X_i) - F_0(X_i)\right)^2 \\ &= \frac{1}{n}\sum \left(\hat{F}(X_{(i)}) - F_0(X_{(i)})\right)^2 = \frac{1}{n}\sum \left(\frac{i}{n} - F_0(X_{(i)})\right)^2\end{aligned}$$

(c) Let $X' = \frac{X-a}{b}$, $b > 0$, then

$$\begin{aligned}\hat{F}'(x) &= \frac{1}{n}\sum I(X_i' \le x) = \frac{1}{n}\sum I((X_i - a)/b \le x) \\ &= \frac{1}{n}\sum I(X_i \le a + bx) = \hat{F}(a + bx) \ne \hat{F}(x)\end{aligned}$$

Therefore none of them is invariant under location and scale.