STAT610 - HWK Solution 10

4.1.5 Let F_{T,θ_0} be the CDF of T under $H : \theta = \theta_0$. By Proposition 4.1.2, the p-value is given by

$$p = \alpha(t) = P_{\theta_0}(T > t) = 1 - F_{T,\theta_0}(t).$$

Since $F_{T,\theta_0}(t) \sim \text{Unif}(0,1)$, $p \sim \text{Unif}(0,1)$.

4.1.6 By 4.1.5, $\alpha(T_1), \cdots, \alpha(T_r) \sim i.i.d.\text{Unif}(0,1)$ iid,

$$\begin{aligned}
&\Rightarrow -\log\alpha(T_1)\cdots - \log\alpha(T_r) \sim Exp(1) \sim \Gamma(1,1) \quad \text{iid} \\
&\Rightarrow \sum_{j=1}^{r}(-2\log\alpha(T_j)) = -2\sum_{j=1}^{r}\log\alpha(T_j) \sim \Gamma(n, 1/2) \sim \chi^2_{2r}
\end{aligned}$$

4.2.5 It is equivalent to do a hypothesis test

$$H : (X,Y) \sim P_0 = N(0,0,1,1,0.6) \quad \text{versus} \quad K : (X,Y) \sim P_1 = N(1,1,1,1,0.6).$$

Suppose $T(X,Y)$ and c minimize $\max\{P_0(T \geq c), P_1(T < c)\}$. Let $\alpha = P_0(T \geq c)$. By the definition of MP test, any size α MP test also minimizes $\max\{P_0(T \geq c), P_1(T < c)\}$.

- First, find an MP level α test. Let

$$L = f(x,y;P_1)/f(x,y;P_0) = \exp\left(\frac{x+y-1}{1+\rho}\right).$$

  By N-P Lemma, $\varphi_k = I(L \geq k) = I(X+Y \geq c)$ with appropriate k and c is an MP size α test.

- Second, determine α such that $\max\{P_0(T \geq c), P_1(T < c)\}$ is minimized. Let $T = X+Y$, $T \sim N(0, 3.2)$ under P_0 and $T \sim N(2, 3.2)$ under P_1,

$$\begin{aligned}
P_0(T \geq c) &= P_0\left(\frac{X+Y}{\sqrt{3.2}} \geq \frac{c}{\sqrt{3.2}}\right) = 1 - \Phi\left(\frac{c}{\sqrt{3.2}}\right) = \Phi\left(-\frac{c}{\sqrt{3.2}}\right) \\
P_1(T < c) &= P_1\left(\frac{X+Y}{\sqrt{3.2}}\right) < \frac{c-2}{\sqrt{3.2}} = \Phi\left(\frac{c-2}{\sqrt{3.2}}\right)
\end{aligned}$$

  $\max\{P_0(T \geq c), P_1(T < c)\}$ is minimized when $\Phi\left(-\frac{c}{\sqrt{3.2}}\right) = \Phi\left(\frac{c-2}{\sqrt{3.2}}\right) \Rightarrow c = 1.$

Therefore, we obtained the optimal rule with statistic $T(X,Y) = X+Y$ and critical value $c = 1$.

4.2.7 Assume φ is an MP level α test, $0 < \alpha < 1$.

- To prove $E_1\varphi(X) \geq \alpha$, consider a randomized test $E_1\delta(X) = \alpha$, in other words, randomly reject H_0 with probability α. It is a size α test, so $E_1\varphi(X) \geq E_1\delta(X) = \alpha$.
- If $E_1\varphi(X) = \alpha$. Then $\delta(X)$ is also an MP test. By N-P Lemma, for $\theta = \theta_0$ and $\theta = \theta_1$,

$$P_\theta[\delta(X) \neq \varphi_k(X), L(X,\theta_0,\theta_1) \neq k] = 0 \tag{1}$$

  where $\varphi_k(X)$ is a size α likelihood ratio test. Since $\delta(X) = \alpha \neq 0$ or 1,

$$P_\theta[\delta(X) = \varphi_k(X), L(X,\theta_0,\theta_1) \neq k] = 0. \tag{2}$$

Combine (1) and (2),

$$P_\theta[L(X,\theta_0,\theta_1) \neq k] = 0$$

Therefore, $p(X,\theta_1) = kp(X,\theta_0)$ with probability 1. By $\int p(x,\theta_1)dx = \int p(x,\theta_0)dx = 1$,

$$k = \int kp(x,\theta_0)dx = \int p(x,\theta_1)dx = 1.$$

- If $p(\cdot,\theta_0) = p(\cdot,\theta_1)$, then $E_1\varphi(X) = E_0\varphi(X) = \alpha$.

4.3.1 (a) $X_1,\cdots,X_n \sim$ Poisson(θ), $p(x;\theta) = \dfrac{e^{n\theta}\theta^{\sum x_i}}{\prod x_i!}$. For $\theta_1 < \theta_2$,

$$\frac{p(x,\theta_2)}{p(x,\theta_1)} = e^{-n(\theta_2-\theta_1)}\left(\frac{\theta_2}{\theta_1}\right)^{\sum x_i}$$

then $p(x,\theta_2)/p(x,\theta_1)$ is increasing in $T = \sum X_i$. This is a MLR family in T. By theorem 4.3.1, A UMP test is of the form $T \geq c$.

(b) Since $T = \sum X_i \sim$ Poisson($n\theta_0$) under $\theta = \theta_0$, $\alpha = P_{\theta_0}(T > c)$. Therefore, α only takes values from $\{P_{\theta_0}(T \geq n) : n = 0,1,2,\cdots\}$.

(c) Need the table of Poisson distribution.