STAT610 - HWK Solution 12

5.3.8 (a) $H:\sigma_1^2=\sigma_2^2(=\sigma^2)$ vs. $H:\sigma_1^2\neq\sigma_2^2$

Under Θ_0, MLEs are $\hat\mu_1=\bar X$, $\hat\mu_2=\bar Y$, $\hat\sigma^2=\frac{\sum(X_i-\bar X)^2+\sum(Y_i-\bar Y)^2}{n_1+n_2}$.

Under Θ, MLEs are $\hat\mu_1=\bar X$, $\hat\mu_2=\bar Y$, $\hat\sigma_1^2=\frac{1}{n_1}\sum(X_i-\bar X)^2$, $\hat\sigma_2^2=\frac{1}{n_2}\sum(Y_i-\bar Y)^2$.

$$\begin{aligned}\lambda&=\frac{\sup_\Theta p(x,y;\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)}{\sup_{\Theta_0}p(x,y;\mu_1,\mu_2,\sigma^2)}=\frac{p(x,y;\hat\mu_1,\hat\mu_2,\hat\sigma_1^2,\hat\sigma_2^2)}{p(x,y;\hat\mu_1,\hat\mu_2,\hat\sigma^2)}\\&=\frac{(\hat\sigma^2)^{\frac{n_1+n_2}{2}}\exp\left(-\frac{\sum(x_i-\bar x)^2}{2\hat\sigma_1^2}-\frac{\sum(y_i-\bar y)^2}{2\hat\sigma_2^2}\right)}{(\hat\sigma_1^2)^{\frac{n_1}{2}}(\hat\sigma_2^2)^{\frac{n_2}{2}}\exp\left(-\frac{\sum(x_i-\bar x)^2+\sum(y_i-\bar y)^2}{2\hat\sigma^2}\right)}\\&\propto\frac{\left(\frac{\sum(x_i-\bar x)^2}{\sum(y_i-\bar y)^2}+1\right)^{\frac{n_1+n_2}{2}}}{\left(\frac{\sum(x_i-\bar x)^2}{\sum(y_i-\bar y)^2}\right)^{\frac{n_1}{2}}}\propto\frac{\left(\frac{s_1^2}{s_2^2}+\frac{n_2-1}{n_1-1}\right)^{\frac{n_1+n_2}{2}}}{\left(\frac{s_1^2}{s_2^2}\right)^{\frac{n_1}{2}}}\end{aligned}$$

(b) Since $(n_1-1)s_1^2/\sigma_1^2\sim\chi^2_{n_1-1}$, $(n_2-1)s_2^2/\sigma_2^2\sim\chi^2_{n_2-1}$, and they are independent,

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}\sim F_{n_1-1,n_2-1}.$$

(c) $P(s_1^2/s_2^2\le c_{k,m})=P\left(\sqrt{\frac{km}{\kappa(k+m)}}\left(\frac{s_1^2}{s_2^2}-1\right)\le z_{1-\alpha}\right)$.

$$\begin{aligned}\sqrt{\frac{km}{\kappa(k+m)}}\left(\frac{s_1^2}{s_2^2}-1\right)&=\sqrt{\frac{km}{\kappa(k+m)}}\left(\frac{(s_1^2-\sigma_1^2)-(s_2^2-\sigma_1^2)}{s_2^2}\right)\\&=\frac{1}{s_2^2}\left(\sqrt{\frac{m}{\kappa(k+m)}}\sqrt{k}(s_1^2-\sigma_1^2)-\sqrt{\frac{k}{\kappa(k+m)}}\sqrt{m}(s_2^2-\sigma_1^2)\right)\end{aligned}\qquad(1)$$

Under H, $E(s_1^2)=E(s_2^2)=\sigma_1^2$. By CLT ans Slusky's Theorem

$$\begin{aligned}\sqrt{k}(s_1^2-\sigma_1^2)&\to\sqrt{n_1}\left(\frac{1}{n_1}\sum[(X_i-\mu)^2-(\bar X-\mu)^2]-\sigma_1^2\right)\\&\to\sigma_1^2\sqrt{n_1}\left(\frac{1}{n_1}\sum\left(\frac{X_i-\mu}{\sigma_1}\right)^2-1\right)-\sqrt{n_1}(\bar X-\mu)^2\\&\to\sigma_1^2N(0,\kappa),\\\text{similarly }\sqrt{m}(s_2^2-\sigma_1^2)&\to\sigma_1^2N(0,\kappa)\end{aligned}$$

Together with $s_2^2\to\sigma_1^2$ by LLN, plug-into (1),

$$\sqrt{\frac{km}{\kappa(k+m)}}\left(\frac{s_1^2}{s_2^2}-1\right)\to\frac{\sigma_1^2}{s_2^2}\left(\sqrt{\frac{m}{\kappa(k+m)}+\frac{k}{\kappa(k+m)}}\right)N(0,\kappa)=N(0,1)$$

Therefore, $P\left(s_1^2/s_2^2\le c_{k,m}\right)\to1-\alpha$.

(d) By LLN, the moment estimate is consistent, as long as the moment exists. Thus, $\hat{c}_{k,m} \to c_{k,m}$, by LLN. Therefore, from (c), $P\left(\frac{s_1^2}{s_2^2} \le \hat{c}_{k,m}\right) \to 1-\alpha$.

5.3.17 $EX = 2(1-\theta)$, $Var(X) = 2\theta(1-\theta)$.

(a) By CLT,

$$\frac{\sqrt{n}(\bar{X}-2(1-\theta))}{\sqrt{2\theta(1-\theta)}} \to N(0,1).$$

$$P(\bar{X} \le t) = P\left(\frac{\sqrt{n}(\bar{X}-2(1-\theta)}{\sqrt{2\theta(1-\theta)}} \le \frac{\sqrt{n}(t-2(1-\theta))}{\sqrt{2\theta(1-\theta)}}\right) \sim \Phi\left(\frac{\sqrt{n}(t-2(1-\theta))}{\sqrt{2\theta(1-\theta)}}\right)$$

(b) By delta method,

$$\frac{\sqrt{n}(\sqrt{\bar{X}}-\sqrt{2(1-\theta)})}{\sqrt{2\theta(1-\theta)}} \to \left[\left.\frac{1}{2\sqrt{x}}\right|_{x=2(1-\theta)}\right] N(0,1)$$

$$\Rightarrow \sqrt{n}\left(\sqrt{\bar{X}}-\sqrt{2(1-\theta)}\right) \to N\left(0,\frac{\theta}{4}\right)$$

(c) By LLN, $\bar{X} \to \mu$. By continuous mapping theorem, $\bar{X}^2 \to \mu^2$. By Slutsky's theorem, $\sqrt{n}(\bar{X}-\mu)+\bar{X}^2 \to N(0,2\theta(1-\theta))+\mu^2 = M(\mu^2, 2\theta(1-\theta))$.

5.3.22 $S_n \sim \Gamma(n/2, 1/2)$. By B.2.4., $E(\sqrt{S_n}) = \frac{\Gamma(1/2+n/2)}{(1/2)^{1/2}\Gamma(n/2)}$.

By Stirling's approximation, $\Gamma(p+1) \sim \sqrt{2\pi}e^{-p}p^{p+\frac{1}{2}}$,

$$\begin{aligned}
E(\sqrt{S_n}) &= \sqrt{2}\Gamma(\frac{n+1}{2})/\Gamma(\frac{n}{2}) \\
&\sim \sqrt{2}\frac{\sqrt{2\pi}\exp\left(-\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right)^{\frac{n}{2}}}{\sqrt{2\pi}\exp\left(-\frac{n-2}{2}\right)\left(\frac{n}{2}-1\right)^{\frac{n-1}{2}}} \\
&= \sqrt{2}e^{-\frac{1}{2}}\left(\frac{n-1}{n-2}\right)^{\frac{n-2}{2}}\left(\frac{n-1}{2}\right)\left(\frac{n-2}{2}\right)^{-1/2} \\
&\sim \frac{n-1}{\sqrt{n-2}} \sim \frac{n+1}{\sqrt{n}} = \sqrt{n}+1/\sqrt{n}
\end{aligned}$$

Hence, $E(\sqrt{S_n}) = \sqrt{n}+R_n$ where $R_n = O(1/\sqrt{n})$.

5.3.28 (a) $E(\bar{Y}-\bar{X}) = \mu_2-\mu_1 = \Delta$, $s^2 = \frac{(n-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$, $T(\Delta) = \sqrt{\frac{n_1 n_2}{n}}(\frac{\bar{Y}-\bar{X}-\Delta}{s})$.
By CLT,

$$\sqrt{\frac{n_1 n_2}{n}}(\bar{X}-\mu_1) = \sqrt{\frac{n_2}{n}}\sqrt{n_1}(\bar{X}-\mu_1) \to \sqrt{(1-\lambda)}N(0,\sigma_1^2) = N(0,(1-\lambda)\sigma_1^2)$$
$$\sqrt{\frac{n_1 n_2}{n}}(\bar{Y}-\mu_2) = \sqrt{\frac{n_1}{n}}\sqrt{n_2}(\bar{Y}-\mu_2) \to \sqrt{\lambda}N(0,\sigma_2^2) = N(0,\lambda\sigma_2^2)$$

Hence,

$$\sqrt{\frac{n_1 n_2}{n}}(\bar{Y}-\bar{X}-\Delta) \to N(0,\lambda\sigma_2^2+(1-\lambda)\sigma_1^2).$$

Since $s^2 \to \lambda\sigma_1^2 + (1-\lambda)\sigma_2^2$, by Slutsky's theorem,

$$T(\Delta) \to N\left(0, \frac{\lambda\sigma_2^2 + (1-\lambda)\sigma_1^2}{\lambda\sigma_1^2 + (1-\lambda)\sigma_2^2}\right) \tag{2}$$

Therefore $P[T(\Delta) \le t] \to \Phi(t\sqrt{\frac{\lambda\sigma_1^2 + (1-\lambda)\sigma_2^2}{\lambda\sigma_2^2 + (1-\lambda)\sigma_1^2}})$.

(b) If $\lambda = \frac{1}{2}$ or $\sigma_1 = \sigma_2$, $T(\Delta) \to N(0,1)$. Also $t_{n-2} \to N(0,1)$. Therefore, (4.9.3) has correct asymptotic probability of coverage.

(c) If $\sigma_2^2 > \sigma_1^2$ and $\lambda > 1-\lambda$, then $T(\Delta) \to N(0,\Sigma)$ in (2) with $\Sigma > 1$. Therefore, (4.9.3) has asymptotic probability of coverage $< 1-\alpha$. Similarly if $\sigma_2^2 < \sigma_1^2$ and $\lambda > 1-\lambda$, then $\Sigma < 1$, and (4.9.3) has asymptotic probability of coverage $> 1-\alpha$.

(d) The asymptotic length of the interval (4.9.3) is given by

$$l_1 = 2t_{n-2}\left(1-\frac{\alpha}{2}\right) s\sqrt{\frac{n}{n_1 n_2}} \to 2\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma_1^2}{1-\lambda} + \frac{\sigma_2^2}{\lambda}} \cdot n^{-\frac{1}{2}}$$

In section (4.9.4), $S_D^2 = \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \to \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$. By CLT and Slutsky's theorem,

$$(\bar{Y} - \bar{X} - \Delta)/S_D \to N(0,1).$$

So the asymptotic length of the interval based on $|D-\Delta|/s_D$ is given by

$$l_2 = 2\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} = 2\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda}} \cdot n^{-\frac{1}{2}}.$$